MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

O. M. BEKETOV NATIONAL UNIVERSITY
of URBAN ECONOMY in KHARKIV

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HIGHER MATHEMATICS

Module 1

LECTURE NOTES

(for full-time and part-time students bachelor education level of the specialty 192 – Construction and civil engineering)

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CONTENT

PREFACE

The purpose of the discipline is to provide a proper fundamental mathematical training of students and to form in them knowledge and ability to apply it for the analysis of a various phenomena according to a diversity spheres of a professional activity. Thus, the task of the discipline is to assist students learn the basics of mathematical apparatus needed to solve theoretical and practical problems, to develop skills and abilities of mathematical research of applied tasks, to develop their analytical and critical thinking, to teach students to understand the scientific sources of professional applications of mathematics.

This syllabus of lectures is designed according to the program of normative educational discipline “Higher mathematics” and the working curriculum of preparation of full-time and part-time students of the “bachelor” education level of the specialty 192 – Construction and civil engineering.

All theoretical material in this lecture notes is structured and coordinated with the classroom lectures conducted during the study of Module 1 topics.

However, this synopsis is not final, because the volume of the studied theoretical material may be changed due to some changes in the curriculum. Therefore, students should follow the classroom lectures carefully and use a wider range of scientific and literary sources, which are presented at the end of the lecture notes, in their preparation for all class.

The lecture notes contain theoretical material as a basic knowledge of the topics of Module 1 that students need to acquire, and self-checking questions.

The lecture notes have a significant number of examples of solving typical tasks, as well as applied tasks, that help student to switch their attention to the practical using of the knowledge to solve professional-oriented tasks.

Some additional information and interest materials are located in the appendices, at the end of the lecture notes.

So many references to sources in which students can find more detailed information about certain mathematical positions or theorems proofs that are not presented in this lecture notes are also given here as an aid to a more in-depth study and search for reference information.

The presented lecture notes will help students to possess the methods of solving practical tasks; it will promote the acquisition of mathematical competencies and intensify students’ independent work.

Students must realize that only active work with lecture notes can help them to be successful in the study of higher mathematics, achieve professional excellence.

CONTENTS OF MODULE 1

Module 1 Linear algebra. Vector and Tensor algebra. Analytic geometry. Differential calculus of one variable functions

Content module 1.1 Linear algebra. Vector and Tensor algebra

Topic 1.1.1 Matrices. Determinants. Systems of linear equations.

Determinants and their properties. Calculation of determinants of any orders.

Matrices and operations with them. Inverse matrix.

Systems of linear algebraic equations. Homogeneous and non-homogeneous systems of linear algebraic equations. Kronecker-Capelli theorem. Solving systems by Cramer's formulas, matrix method, Gaussian method.

Topic 1.1.2 Vectors. Tensors.

Scalar and vector values. The concept of vector. Conditions of a vectors equality . Linear operations with vectors. Decomposition of the vector on the basis of coordinate orts. Linear operations with vectors given by their coordinates.

Scalar, vector and mixed products of vectors. Vector magnitude, angle between vectors, guide cosines. Conditions of collinearity, orthogonality and coplanarity of vectors. Geometric applications of products of vectors. Coordinates of the vector on this basis.

Basic concepts of tensor calculus. Recording of tensor expressions. Convolution. Tensor invariant. Metric tensor.

Content module 1.2 Analytic geometry.

Topic 1.2.1 Elements of analytical geometry on the plane and in a space.

Straight line on the plane. Cartesian rectangular coordinate system on the plane. The distance between two points. Dividing a segment in a given ratio. The main types of equations of the line on the plane. The angle between the lines. Conditions of parallelty and perpendicularity of lines. Distance from point to line.

Basic types of equations of a plane and a line in a space. Angles between lines and planes. Conditions of parallelity and perpendicularity. The distance from the point to the plane.

Topic 1.2.2 Second order curves and surfaces.

Second order curves. Equation of a circle with a given center and radius. Canonical equations of a circle, an ellipse, a hyperbola and a parabola. Investigation of their forms.

Surfaces of the second order. Cylindrical surfaces. Conical surfaces. Second order cone. Surface rotation. Sphere. Ellipsoid. Hyperboloids. Paraboloids.

Polar coordinate system. Parametric form of lines.

Content module 1.3 Differential calculus of one variable functions.

Topic 1.3.1 Limits. Derivative. Differential.

Limits theory. Variables and constant values. infinitesimal and infinitude variables and their properties. Variable limit. Properties of limits. The first and second standard limits. Comparison of infinitesimals. The equivalents are infinitesimal. Indeterminate forms and their disclosure.

Function. Continuity. The concept of function. Ways to set the function. Basic elementary functions and their graphs.

The concept of derivative as the velocity of a function change. Geometric sense of the derivative. Tangent and normal to the graph of the function. The physical sense of the derivative. Derivative properties. Basic rules of differentiation. Table of derivatives.

The original composite function. Derivatives of implicit and inverse functions. Rule of logarithmic differentiation. Derivative of parametrically defined function. Derivatives of higher orders. The physical sense of the second derivative.

Function differential. Properties of differential. The relationship between the differential and the derivative. Derivatives and differentials of higher orders.

Topic 1.3.2 The application of a derivative.

L'Hospital's rule for evaluating indeterminate forms.

Conditions for increasing and decreasing the function. Extremes of the function. The smallest and largest value of the function on the segment

Conditions of convexity and concavity of the graph of the function and the presence of inflection. Asymptotes of the graph of the function.

The general scheme of a function research and construction of its graph.

Lecture 1
Determinants.
Cramer's Rule for solving the systems of linear algebraic equations

Determinant is a scalar value that can be computed from the elements of a square table. The ***element of determinant*** is a number denoted as a_{ij}, where indices i and j indicate the location of this element in the table of numbers:

i is a number of a row,

j is a number of a column.

The determinant is denoted Δ_n $(\det)$, where index n indicates the order of the determinant which is the number of its rows (columns).

For example, $\begin{vmatrix} -1 & 3 \\ 7 & 0 \end{vmatrix}$ is the determinant of second order, because there are two rows and two columns, so it can be marked as: $\Delta_2 = \begin{vmatrix} -1 & 3 \\ 7 & 0 \end{vmatrix}$.

If we say that the number 3 is the element of a determinant Δ_2, which is located at the first row and the second column, we can denote it as: $a_{12} = 3$.

$$\Delta_3 = \begin{vmatrix} 3 & -2 & 6 \\ 5 & 1 & 0 \\ -2 & -7 & 3 \end{vmatrix}$$ is the determinant of the third order.

The ***main diagonal*** of the determinant is the diagonal, which consists of elements $a_{11}, a_{22}, a_{33}, ..., a_{nn}$. Another diagonal is called a secondary diagonal.

The second-order determinant is calculated by the "cross" rule:

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

The third-order determinant is calculated by “asterisk rule” (“rule of triangles”, Sarrus’ rule or Sarrus’ scheme):

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{21} \cdot a_{32} \cdot a_{13} -$$

$$- a_{13} \cdot a_{22} \cdot a_{31} - a_{12} \cdot a_{21} \cdot a_{33} - a_{32} \cdot a_{23} \cdot a_{11}..$$

Schematically these rules can be represented as it is shown on the Figures 1.1, 1.2.

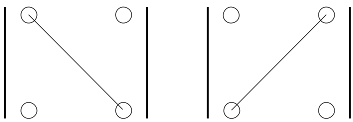

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Figure 1.1

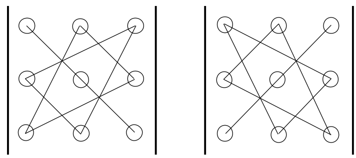

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Figure 1.2

Example 1.1 Find the determinants of the matrices A and B:

$$A = \begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 3 & 4 \\ 3 & -2 & -2 \end{pmatrix}.$$

Solution:

$$\det A = \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = 3 \cdot 2 - 4 \cdot (-1) = 6 + 4 = 10,$$

$$\det B = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 4 \\ 3 & -2 & -2 \end{vmatrix} = -6-12-12-27-4+8=-53 .$$

The ***minor*** M_{ij} of the entry a_{ij} in the i th row and j th is the determinant of the submatrix formed by deleting the i th row and j th column. The ***cofactor*** A_{ij} is obtained by multiplying the minor and $(-1)^{i+j}$:

$$A_{ij} = (-1)^{i+j} M_{ij} .$$

Example 1.2 Find M_{12} and A_{12}:

$$\begin{vmatrix} 2 & 0 & -1 \\ 3 & 1 & -2 \\ -5 & 2 & 3 \end{vmatrix} .$$

Solution:

$$M_{12} = \begin{vmatrix} 3 & -2 \\ -5 & 3 \end{vmatrix} = 9-10 = -1 ,$$

$$A_{12} = (-1)^{1+2} \cdot M_{12} = -1 \cdot (-1) = 1 .$$

The determinant of the n ***th order*** is equal to the sum of n products of the elements of the i th row or j th column on their cofactors (cofactor expansion along the i th row or j th column):

$$\Delta_n = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} a_{ij} A_{ij} .$$

Example 1.3 Using a cofactor expansion along the first row compute the determinant

$$\begin{vmatrix} 1 & 8 & 3 \\ 3 & 6 & 1 \\ 2 & 6 & 2 \end{vmatrix}.$$

Solution:

$$\begin{vmatrix} 1 & 8 & 3 \\ 3 & 6 & 1 \\ 2 & 6 & 2 \end{vmatrix} = 1\cdot(-1)^{1+1}\cdot\begin{vmatrix} 6 & 1 \\ 6 & 2 \end{vmatrix} + 8\cdot(-1)^{1+2}\cdot\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + 3\cdot(-1)^{1+3}\cdot\begin{vmatrix} 3 & 6 \\ 2 & 6 \end{vmatrix} =$$

$$= 12-6-8\cdot(6-2)+3\cdot(18-12) = 6-8\cdot 4+3\cdot 6 = 6-32+18 = -8.$$

Note. Elements of the determinant may be not only numbers, but other objects as well.

Example 1.4 Solve the equation $\begin{vmatrix} x & -2 & 2 \\ 1 & x & -1 \\ 1 & -1 & 1 \end{vmatrix} = 0$.

Solution. We will use the rule of triangles:

$$\begin{vmatrix} x & -2 & 2 \\ 1 & x & -1 \\ 1 & -1 & 1 \end{vmatrix} = x^2+2-2-2x+2-x = x^2-3x+2, \quad x^2-3x+2=0,$$

$$D = b^2-4ac = 9-4\cdot 2 = 1,$$

$$x_{1,2} = \frac{-b\pm\sqrt{D}}{2a} = \frac{3\pm 1}{2},$$

$$x_1 = 2,\ x_2 = 1.$$

Now let’s go on to the topic how the concept of determinant is used to solve systems of linear algebraic equations, in particular we will consider Cramer’s rule for solving the systems of linear algebraic equations. Let us consider a system of n linear equations with n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1; \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n. \end{cases} \tag{1.1}$$

For such systems we can use ***Cramer's Rule***:

if $\Delta = \det A = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \neq 0$, then the system (1.1) has the unique solution $x_j = \dfrac{\Delta_j}{\Delta}$ $(j = \overline{1,n})$, where the determinant Δ_j is formed from Δ by replacing column j with the vector B of constants.

Cramer's Rule lets us by-eye solve systems that are small and simple. For example, we can solve systems with two equations and two unknowns, or three equations and three unknowns, where the numbers are small integers. Such cases appear often enough that many people find this formula handy. But using it to solving large or complex systems is not practical, either by hand or by a computer.

If $\Delta = 0$, then two cases are possible:

1) the system is incompatible, i.e. it has no solution if at least one of $\Delta_j \neq 0$;

2) the system is indeterminate, i.e. it has an infinite number of solutions if all $\Delta_j = 0$.

Example 1.5 Solve the system of equations by the Cramer's Rule:

$$\begin{cases} x+2y+3z=8; \\ 3x+y+z=6; \\ 2x+y+2z=6. \end{cases}$$

*Solution. A*s we have three unknowns, we should calculate four determinants. Start to calculate the main determinant of the given system consisted of the coefficients of all equations

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = 2+9+4-6-12-1 = -4 \neq 0,$$

$$\Delta_1 = \begin{vmatrix} 8 & 2 & 3 \\ 6 & 1 & 1 \\ 6 & 1 & 2 \end{vmatrix} = 16+18+12-18-24-8 = -4,$$

$$\Delta_2 = \begin{vmatrix} 1 & 8 & 3 \\ 3 & 6 & 1 \\ 2 & 6 & 2 \end{vmatrix} = 12+54+16-36-48-6 = -8,$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 8 \\ 3 & 1 & 6 \\ 2 & 1 & 6 \end{vmatrix} = 6+24+24-16-6-36 = -4.$$

Thus, the solution to the system has the following form:

$$x = \frac{-4}{-4} = 1, \quad y = \frac{-8}{-4} = 2, \quad z = \frac{-4}{-4} = 1.$$

Let us check the obtained solution:

$$\begin{cases} 1+2\cdot 2+3\cdot 1 = 1+4+3 = 8; \\ 3\cdot 1+2+1 = 3+2+1 = 6; \\ 2\cdot 1+2+2\cdot 1 = 2+2+2 = 6. \end{cases}$$

Answer: $x=1$, $y=2$, $z=1$.

Lecture 2
Matrices

A ***matrix*** is a system of elements (in the particular case of numbers) arranged in a certain order and forming a table. If in this table there are m rows and n columns, and its elements (entries) are denoted by a_{ij}, where $i = \overline{1,m}$ is the row number, and $j = \overline{1,n}$ is the column number, at the intersection of which this element is located, then the matrix is written in the following form:

$$A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

or abbreviated $A_{m\times n} = \left(a_{ij}\right)$.

The matrix, all elements of which are equal to zero, is called a ***zero matrix*** and is denoted by O.

The matrix, in which the elements of the main diagonal are equal to one, and all the rest ones are zero, is called the ***identity matrix*** and is denoted by E.

The matrix, which consists of only one row, is called a ***row vector***. The matrix, which consists of only one column is called a ***column vector***.

The matrix A^T obtained from the matrix A by replacing each row with a column of the same number is called the ***transposed matrix***.

Example 2.1 Find the transposed matrix A^T for

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 3 \end{pmatrix}.$$

Solution. Do it

$$A^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 3 \end{pmatrix}.$$

Two matrices A and B are called ***equal*** if:

1) they have the same size;

2) all the corresponding elements of the matrices are equal to each other, i.e. $a_{ij} = b_{ij}$ ($i = \overline{1,m}$, $j = \overline{1,n}$).

If $m = n$, then the matrix is called the ***square matrix of the*** n ***th order***. In a square matrix, the elements $a_{11}, a_{22}, ..., a_{nn}$ for which $i = j$ form the ***main diagonal*** of the matrix.

A square matrix is called ***lower triangular*** matrix if all the entries above the main diagonal are zero. Similarly, a square matrix is called ***upper triangular*** matrix if all the entries below the main diagonal are zero.

Consider the operation with matrices.

The ***sum*** of two rectangular matrices A and B of equal sizes ($m \times n$) is the matrix C of the same size, whose elements $c_{i\,j}$ are equal to the sum of the corresponding elements of the matrices A and B, i.e. $c_{i\,j} = a_{i\,j} + b_{i\,j}$ ($i = \overline{1,m}$, $j = \overline{1,n}$).

Example 2.2 Find the sum of the matrices

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 & 0 \\ 3 & -3 & 1 \end{pmatrix}.$$

Solution:

$$A + B = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 4 & 0 \\ 3 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1+2 & 3+4 & -1+0 \\ 2+3 & 4+(-3) & 3+1 \end{pmatrix} = \begin{pmatrix} 3 & 7 & -1 \\ 5 & 1 & 4 \end{pmatrix}.$$

The ***subtraction*** of two rectangular matrices A and B of equal sizes ($m \times n$) is the matrix D of the same size, whose elements d_{ij} are equal to the subtraction of the corresponding elements of the matrices A and B, i.e. $d_{ij} = a_{ij} - b_{ij}$ $(i = \overline{1,m}$, $j = \overline{1,n})$.

Example 2.3 Find the subtraction $A - B$ and $B - A$ of the matrices

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 3 \end{pmatrix},\ B = \begin{pmatrix} 2 & 4 & 0 \\ 3 & -3 & 1 \end{pmatrix}.$$

Solution:

$$A-B=\begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 3 \end{pmatrix}-\begin{pmatrix} 2 & 4 & 0 \\ 3 & -3 & 1 \end{pmatrix}=\begin{pmatrix} 1-2 & 3-4 & -1-0 \\ 2-3 & 4-(-3) & 3-1 \end{pmatrix}=\begin{pmatrix} -1 & -1 & -1 \\ -1 & 7 & 2 \end{pmatrix},$$

$$B-A=\begin{pmatrix} 2 & 4 & 0 \\ 3 & -3 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 3 \end{pmatrix}=\begin{pmatrix} 2-1 & 4-3 & 0-(-1) \\ 3-2 & -3-4 & 1-3 \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 \\ 1 & -7 & -2 \end{pmatrix}.$$

The ***multiplication of the matrix*** A ***by the number*** m is the matrix B, which is obtained by multiplying the number m by each element of the matrix A, i.e. $b_{ij} = m \cdot a_{ij}$ $(i = \overline{1,m}$, $j = \overline{1,n})$.

Example 2.4 Multiply the matrix $A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 3 \end{pmatrix}$ by 2.

Solution:

$$2A = 2 \cdot \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 & 2\cdot 3 & 2\cdot(-1) \\ 2\cdot 2 & 2\cdot 4 & 2\cdot 3 \end{pmatrix} = \begin{pmatrix} 2 & 6 & -2 \\ 4 & 8 & 6 \end{pmatrix}.$$

The ***product*** of the matrix $A_{m\times p}$ with the matrix $B_{p\times n}$ is the matrix $C_{m\times n}$, such that the element c_{ij} of the matrix C, standing in the i th row and j th column, is equal to the sum of the products of the i th row of the matrix A by the corresponding elements of the j th column of the matrix B, that is

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{im}b_{mj} \quad (i = \overline{1,m}\,,\ j = \overline{1,n}\,).$$

The product of matrices exists if and only if when the number of columns of the first matrix is equal to the number of rows of the second matrix. Note that matrix multiplication is not commutative: BA is usually not equal to AB.

Example 2.5 Find the products AB and BA of the matrices

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -3 \\ 3 & 0 \\ 4 & 2 \end{pmatrix}.$$

Solution:

$$AB = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 \\ 3 & 0 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 2 + 3\cdot 3 - 1\cdot 4 & 1\cdot(-3) + 3\cdot 0 - 1\cdot 2 \\ 2\cdot 2 + 4\cdot 3 + 3\cdot 4 & 2\cdot(-3) + 4\cdot 0 + 3\cdot 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 2+9-4 & -3+0-2 \\ 4+12+12 & -6+0+6 \end{pmatrix} = \begin{pmatrix} 7 & -5 \\ 28 & 0 \end{pmatrix},$$

$$BA = \begin{pmatrix} 2 & -3 \\ 3 & 0 \\ 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 - 3\cdot 2 & 2\cdot 3 - 3\cdot 4 & 2\cdot(-1) - 3\cdot 3 \\ 3\cdot 1 + 0\cdot 2 & 3\cdot 3 + 0\cdot 4 & 3\cdot(-1) + 0\cdot 3 \\ 4\cdot 1 + 2\cdot 2 & 4\cdot 3 + 2\cdot 4 & 4\cdot(-1) + 2\cdot 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 2-6 & 6-12 & -2-9 \\ 3+0 & 9+0 & -3+0 \\ 4+4 & 12+8 & -4+6 \end{pmatrix} = \begin{pmatrix} -4 & -6 & -11 \\ 3 & 9 & -3 \\ 8 & 20 & 2 \end{pmatrix}.$$

An important concept is the inverse matrix.

A square matrix A is ***invertible*** if and only if its determinant is not equal to zero. A matrix A^{-1} is called ***inverse*** to the invertible square matrix A if the condition $AA^{-1} = A^{-1}A = E$ is fulfilled.

Algorithm of the inverse matrix finding:

1. Calculate the determinant of the matrix A.
2. Find the transposed matrix A^T.
3. Find the cofactors for each element of the matrix A^T and write down the inverse matrix $A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} A_{11}^T & A_{12}^T & \dots & A_{1n}^T \\ A_{21}^T & A_{22}^T & \dots & A_{2n}^T \\ \dots & \dots & \dots & \dots \\ A_{n1}^T & A_{n2}^T & \dots & A_{nn}^T \end{pmatrix}$.
4. Check the condition $AA^{-1} = A^{-1}A = E$.

Example 2.6. Find the inverse matrix A^{-1} for matrix

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 3 & 1 \\ 2 & 3 & 4 \end{pmatrix}.$$

Solution.

1. Let us calculate the determinant of the matrix A:

$$\det A = \begin{vmatrix} 3 & 0 & 1 \\ 1 & 3 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 36 + 3 + 0 - 6 - 0 - 9 = 24 \neq 0.$$

2. Let us find the transposed matrix: $A^T = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 3 \\ 1 & 1 & 4 \end{pmatrix}$.

3. Let us find the cofactors for each element of the matrix A^T:

$$A_{11} = (-1)^2 \begin{vmatrix} 3 & 3 \\ 1 & 4 \end{vmatrix} = 12 - 3 = 9, \quad A_{12} = (-1)^3 \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} = -(0-3) = 3,$$

$$A_{13} = (-1)^4 \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = 0 - 3 = -3, \quad A_{21} = (-1)^3 \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -(4-2) = -2,$$

$$A_{22}=(-1)^4\begin{vmatrix}3&2\\1&4\end{vmatrix}=12-2=10,\quad A_{23}=(-1)^5\begin{vmatrix}3&1\\1&1\end{vmatrix}=-(3-1)=-2,$$

$$A_{31}=(-1)^4\begin{vmatrix}1&2\\3&3\end{vmatrix}=3-6=-3,\quad A_{32}=(-1)^5\begin{vmatrix}3&2\\0&3\end{vmatrix}=-(9-0)=-9,$$

$$A_{33}=(-1)^6\begin{vmatrix}3&1\\0&3\end{vmatrix}=9-0=9.$$

Thus, the inverse matrix has the following form:

$$A^{-1}=\frac{1}{24}\begin{pmatrix}9&3&-3\\-2&10&-2\\-3&-9&9\end{pmatrix}.$$

4. Let us check the condition $AA^{-1}=A^{-1}A=E$:

$$A^{-1}A=\frac{1}{24}\begin{pmatrix}9&3&-3\\-2&10&-2\\-3&-9&9\end{pmatrix}\cdot\begin{pmatrix}3&0&1\\1&3&1\\2&3&4\end{pmatrix}=$$

$$=\frac{1}{24}\begin{pmatrix}9\cdot3+3\cdot1-3\cdot2&9\cdot0+3\cdot3-3\cdot3&9\cdot1+3\cdot1-3\cdot4\\-2\cdot3+10\cdot1-2\cdot2&-2\cdot0+10\cdot3-2\cdot3&-2\cdot1+10\cdot1-2\cdot4\\-3\cdot3-9\cdot1+9\cdot2&-3\cdot0-9\cdot3+9\cdot3&-3\cdot1-9\cdot1+9\cdot4\end{pmatrix}=$$

$$=\frac{1}{24}\begin{pmatrix}27+3-6&0+9-9&9+3-12\\-6+10-4&0+30-6&-2+10-8\\-9-9+18&0-27+27&-3-9+36\end{pmatrix}=\frac{1}{24}\begin{pmatrix}24&0&0\\0&24&0\\0&0&24\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=E.$$

Answer: $A^{-1}=\dfrac{1}{24}\begin{pmatrix}9&3&-3\\-2&10&-2\\-3&-9&9\end{pmatrix}.$

Lecture 3
Inverse matrix method and Gaussian Elimination method for solving the linear algebraic equations systems

A ***system of linear algebraic equations*** (***SLAE***) consisting of m equations with n unknowns is a system of the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1; \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m, \end{cases} \tag{3.1}$$

where x_j are the unknowns, a_{ij} are the coefficients of the system, b_i are the constant terms ($i = \overline{1, m}$, $j = \overline{1, n}$).

Such systems (3.1) are conveniently written in a ***matrix form***:

$$A \cdot X = B,$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$ is the ***coefficient matrix***,

$X = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$ is the ***solution vector***, $B = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{pmatrix}$ is the ***vector of constants***.

The ***augmented matrix*** of the system of equations is the matrix whose first n columns are the columns of matrix A and whose last $(n+1)$ column is the column vector B:

$$\overline{A} = \left(A|B\right) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right).$$

The ***solution*** to a system is a set of n numbers that, when substituted instead of the unknowns in the equations, turn all the equations of the system into identities.

A system of equations is called ***compatible*** if it has at least one solution, and ***incompatible*** if it has no solutions.

A consistent system is called ***independent*** if it has exactly one solution, and ***dependent*** if it has more than one solution.

A system is said to be ***homogeneous*** if all its constant terms are zero $b_i = 0$ ($i = \overline{1, m}$), and ***non-homogeneous*** if at least one of the constant terms is non-zero.

A homogeneous system of equations is always consistent, since there is always a ***trivial solution***: $x_1 = x_2 = \ldots = x_n = 0$. If the determinant of a homogeneous system is nonzero ($\Delta \neq 0$), then the system has a unique zero solution. If $\Delta = 0$, then a homogeneous system has an infinite number of solutions.

Let us consider a system of n linear equations with n unknowns, written in matrix form:

$$A \cdot X = B. \tag{3.2}$$

If the matrix A is invertible ($\det A \neq 0$), then it has an inverse matrix A^{-1}. Multiplying from the left both sides of the equation (3.2) by A^{-1}, we get the solution of this system:

$$A^{-1} \cdot A \cdot X = A^{-1} \cdot B, \quad E \cdot X = A^{-1} \cdot B,$$

$$X = A^{-1} \cdot B.$$

Example 3.1 Solve a system of equations using the inverse matrix method:

$$\begin{cases} x+y+z=2; \\ 5x-4y-4z=1; \\ 2x+y+2z=2. \end{cases}$$

Solution. Write down the matrices:

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 5 & -4 & -4 \\ 2 & 1 & 2 \end{pmatrix},\ X=\begin{pmatrix} x \\ y \\ z \end{pmatrix},\ B=\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Firstly, we should find the inverse matrix:

$$\det A=\begin{vmatrix} 1 & 1 & 1 \\ 5 & -4 & -4 \\ 2 & 1 & 2 \end{vmatrix}=-8+5-8+8-10+4=-9\neq 0,$$

$$A^T=\begin{pmatrix} 1 & 5 & 2 \\ 1 & -4 & 1 \\ 1 & -4 & 2 \end{pmatrix},$$

$$A_{11}=(-1)^2\begin{vmatrix} -4 & 1 \\ -4 & 2 \end{vmatrix}=-8+4=-4,\quad A_{12}=(-1)^3\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}=-\left(2-1\right)=-1,$$

$$A_{13}=(-1)^4\begin{vmatrix} 1 & -4 \\ 1 & -4 \end{vmatrix}=-4+4=0,$$

$$A_{21}=(-1)^3\begin{vmatrix} 5 & 2 \\ -4 & 2 \end{vmatrix}=-\left(10+8\right)=-18,$$

$$A_{22}=(-1)^4\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}=2-2=0,\quad A_{23}=(-1)^5\begin{vmatrix} 1 & 5 \\ 1 & -4 \end{vmatrix}=-\left(-4-5\right)=9,$$

$$A_{31}=(-1)^4\begin{vmatrix} 5 & 2 \\ -4 & 1 \end{vmatrix}=5+8=13,\quad A_{32}=(-1)^5\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}=-\left(1-2\right)=1,$$

$$A_{33}=(-1)^6\begin{vmatrix}1 & 5\\ 1 & -4\end{vmatrix}=-4-5=-9,\quad A^{-1}=-\frac{1}{9}\begin{pmatrix}-4 & -1 & 0\\ -18 & 0 & 9\\ 13 & 1 & -9\end{pmatrix},$$

$$A^{-1}A=-\frac{1}{9}\begin{pmatrix}-4 & -1 & 0\\ -18 & 0 & 9\\ 13 & 1 & -9\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ 5 & -4 & -4\\ 2 & 1 & 2\end{pmatrix}=$$

$$=-\frac{1}{9}\begin{pmatrix}-4-5+0 & -4+4+0 & -4+4+0\\ -18+0+18 & -18+0+9 & -18+0+18\\ 13+5-18 & 13-4-9 & 13-4-18\end{pmatrix}=$$

$$=-\frac{1}{9}\begin{pmatrix}-9 & 0 & 0\\ 0 & -9 & 0\\ 0 & 0 & -9\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Secondly, we should find the solution to our system by the formula:

$$X=A^{-1}\cdot B=-\frac{1}{9}\begin{pmatrix}-4 & -1 & 0\\ -18 & 0 & 9\\ 13 & 1 & -9\end{pmatrix}\cdot\begin{pmatrix}2\\ 1\\ 2\end{pmatrix}=-\frac{1}{9}\begin{pmatrix}-4\cdot 2-1\cdot 1+0\cdot 2\\ -18\cdot 2+0\cdot 1+9\cdot 2\\ 13\cdot 2+1\cdot 1-9\cdot 2\end{pmatrix}=-\frac{1}{9}\begin{pmatrix}-9\\ -18\\ 9\end{pmatrix}=\begin{pmatrix}1\\ 2\\ -1\end{pmatrix}.$$

Let us check the obtained solution:

$$\begin{cases}1+2-1=2;\\ 5\cdot 1-4\cdot 2-4\cdot(-1)=5-8+4=1;\\ 2\cdot 1+2+2\cdot(-1)=2+2-2=2.\end{cases}$$

Answer: $x=1$, $y=2$, $z=-1$.

The next method we will consider will be **Gaussian Elimination** method.

Consider a system of m equations with n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1; \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m. \end{cases}$$

Let us define certain operations on matrices called ***elementary row and column operations***.

1. Interchanging (swapping) the i th and j th rows (columns), where $i \neq j$.

2. Multiplying the i th row by a non-zero quantity.

3. Adding a multiple of the j th row to the i th one, where $i \neq j$. Note, that you should leave the first row the same after this operation, but replace the second row by the new values.

These operations allow us to obtain equivalent systems to the initial one, but with a form that simplifies obtaining the solution.

Algorithm of applying the Gaussian Elimination method.

1. Construct the augmented matrix for the system.

2. Use elementary row and column operations to transform the augmented matrix into a triangular one.

3. Write down the new linear system for which the triangular matrix is the associated augmented matrix.

4. Solve the new system starting from the last equation. You may need to assign some parametric values to some unknowns. Then apply the method of back substitution to solve the new system.

Example 3.2 Solve the following system via Gaussian elimination:

$$\begin{cases} x_1 + x_2 + 2x_3 = -1; \\ 2x_1 - x_2 + 2x_3 = -4; \\ 4x_1 + x_2 + 4x_3 = -2. \end{cases}$$

Solution. Write down the augmented matrix of the system:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 2 & -1 & 2 & -4 \\ 4 & 1 & 4 & -2 \end{array}\right).$$

Multiply the first row by (-2) and add to the second one, then by (-4) and add to the third row: $\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -2 & -2 \\ 0 & -3 & -4 & 2 \end{array}\right)$.

Further, multiply the second row by (-1) and add to the third one:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -2 & -2 \\ 0 & 0 & -2 & 4 \end{array}\right).$$

Write down the new linear system: $\begin{cases} x_1 + x_2 + 2x_3 = -1; \\ -3x_2 - 2x_3 = -2; \\ -2x_3 = 4. \end{cases}$

From the third equation we find: $x_3 = -2$.

Further, we substitute the x_3 into the second equation and get:

$$-3x_2 - 2\cdot(-2) = -2, \quad -3x_2 + 4 = -2, \quad -3x_2 = -6, \quad x_2 = 2.$$

Finally, we substitute the x_3 and x_2 into the first equation:

$$x_1 + 2 + 2\cdot(-2) = -1, \quad x_1 + 2 - 4 = -1, \quad x_1 = 1.$$

Let us check the obtained solution:

$$\begin{cases} 1+2+2\cdot(-2)=1+2-4=-1; \\ 2\cdot 1-2+2\cdot(-2)=2-2-4=-4; \\ 4\cdot 1+2+4\cdot(-2)=4+2-8=-2. \end{cases}$$

Answer: $x_1 = 1$, $x_2 = 2$, $x_3 = -2$.

Example 3.3 Solve the homogeneous system of equations:

$$\begin{cases} x-y+3z=0; \\ 4x+2y-z=0; \\ x+5y-10z=0. \end{cases}$$

Solution. Write down the augmented matrix of the system:

$$\begin{pmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ 1 & 5 & -10 \end{pmatrix}.$$

Multiply the first row by (-4) and add to the second one, then by (-1) and add to the third row:

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 6 & -13 \\ 0 & 6 & -13 \end{pmatrix}.$$

Multiply the second row by (-1) and add to the third one:

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 6 & -13 \\ 0 & 0 & 0 \end{pmatrix}.$$

Write down the new linear system:

$$\begin{cases} x - y + 3z = 0; \\ \quad 6y - 13z = 0. \end{cases}$$

Set $z = t$, $t \in R$, then we have:

$$\begin{cases} x - y + 3t = 0; \\ \quad 6y - 13t = 0. \end{cases}$$

From the second equation we find:

$$6y = 13t, \quad y = \frac{13}{6}t.$$

Further, we substitute y into the first equation and get:

$$x - \frac{13}{6}t + 3t = 0, \quad x = \frac{13}{6}t - 3t = \frac{13-18}{6}t = -\frac{5}{6}t.$$

Set $t = 6k$, $k \in R$, then the solution to the system will take the form:

$$\begin{cases} x = -5k; \\ y = 13k; \\ z = 6k. \end{cases}$$

Let us check the obtained solution:

$$\begin{cases} -5k - 13k + 3 \cdot 6k = -5k - 13k + 18k = 0; \\ 4 \cdot (-5k) + 2 \cdot 13k - 6k = -20k + 26k - 6k = 0; \\ -5k + 5 \cdot 13k - 10 \cdot 6k = -5k + 65k - 60k = 0. \end{cases}$$

Answer: $x = -5k$, $y = 13k$, $z = 6k$, $k \in R$.

Lecture 4
Vectors algebra

Many physical quantities, such as mass, time, temperature are fully specified by one value (magnitude) which is a real number. Such quantities are called ***scalars***. But other quantities such as speed, force, electric field intensity require more than one value to describe them. They are ***vectors***.

A directed line segment is a ***vector***, denoted as $\overrightarrow{AB}$ or as $\vec{a}$, and read as "vector $\overrightarrow{AB}$" or "vector $\vec{a}$".

The point A from where the vector $\overrightarrow{AB}$ starts is called its ***initial point***, and the point B where it ends is called its ***terminal point***. The distance between initial and terminal points of a vector is called the ***magnitude*** (or ***length***) of the vector, denoted as $\left|\overrightarrow{AB}\right|$ or $|\vec{a}|$.

A vector whose initial and terminal points coincide, is called a ***zero vector*** (or ***null vector***), and denoted as $\vec{0}$. Zero vector can not be assigned a definite direction as it has zero magnitude.

A vector whose magnitude is unity is called a ***unit vector***.

Two or more vectors are said to be ***collinear*** $\vec{a} \parallel \vec{b}$ if they are parallel to the same line, irrespective of their magnitudes and directions.

Two vectors are said to be ***equal***, if they have the same magnitude and direction regardless of the positions of their initial points, and written as $\vec{a} = \vec{b}$.

A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{AB}$), but direction is opposite to that of it, is called ***negative*** of the given vector. For example, vector $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$, and written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Vectors that locate on parallel planes or in the same plane are called ***coplanar***.

Let's consider the linear vector operations.

The ***sum*** $\vec{a}+\vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ is a vector, which is determined by the triangle rule (Figure 4.1) or by the parallelogram rule (Figure 4.2).

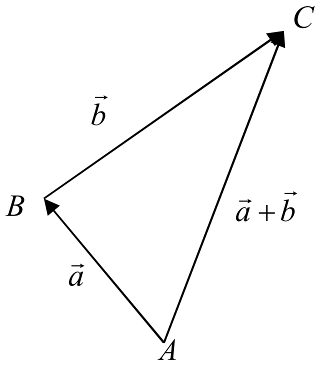


Figure 4.1

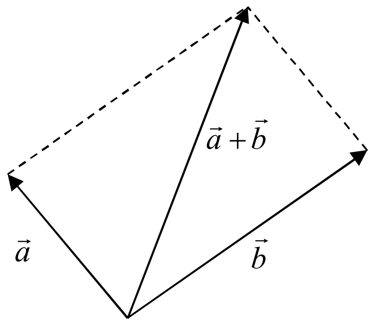


Figure 4.2

According to the ***triangle law of vector addition*** two vectors $\vec{a}$ and $\vec{b}$ are positioned so that the initial point of one coincides with the terminal point of the other. Then, the vector $\vec{a}+\vec{b}$ is third side AC of the triangle ABC (Figure 4.1).

According to the ***parallelogram law of vector addition*** $\vec{a}+\vec{b}$ is the diagonal of the parallelogram formed by vectors $\vec{a}$ and $\vec{b}$ (Figure 4.2).

The ***difference*** $\vec{a}-\vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ is a vector, which in sum with the vector $\vec{b}$ gives a vector $\vec{a}$:

$$\vec{a}-\vec{b}=\vec{a}+\left(-\vec{b}\right).$$

The ***product of the vector*** $\vec{a}$ ***by the scalar*** λ, denoted as $\lambda\vec{a}$, is called a vector $\lambda\vec{a}$, collinear to the vector $\vec{a}$. It has the direction same (or opposite) to that of vector $\vec{a}$ according as the value of λ is positive (or negative). Also, the magnitude of vector $\lambda\vec{a}$ is $|\lambda|$

times the magnitude of the vector $\vec{a}$, i.e.,

$$|\lambda\vec{a}| = |\lambda| \cdot |\vec{a}|.$$

The vector $-\vec{a}$ is called the ***negative*** (or ***additive inverse***) ***of vector*** $\vec{a}$ and we always have

$$\vec{a} + (-\vec{a}) = \vec{0}.$$

The considered operations are called linear, since have the appropriate properties (similar to the properties of operations on real numbers):

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}, \quad \vec{a} + \vec{b} + \vec{c} = \left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right),$$

$$\lambda\vec{a} = \vec{a}\lambda, \quad (\alpha\beta)\vec{a} = \alpha(\beta\vec{a}), \quad \lambda\left(\vec{a} + \vec{b}\right) = \lambda\vec{a} + \lambda\vec{b},$$

$$(\alpha + \beta)\vec{a} = \alpha\vec{a} + \beta\vec{a}, \quad \vec{a} + \vec{0} = \vec{a}, \quad 1\vec{a} = \vec{a}.$$

Let us take the points $A(1,0,0)$, $B(0,0,1)$ and $C(0,0,1)$ on the x-axis, y-axis and z-axis, respectively. Then, clearly

$$\left|\overrightarrow{OA}\right| = 1, \ \left|\overrightarrow{OB}\right| = 1, \ \left|\overrightarrow{OC}\right| = 1.$$

The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, each having magnitude 1, are called ***unit vectors along the axes*** Ox, Oy and Oz, respectively, and denoted by $\vec{i}$, $\vec{j}$ and $\vec{k}$, respectively.

The position vector of any point $M(x, y, z)$ with reference to the origin is given by its ***component form***:

$$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k},$$

where x, y and z are called as the ***scalar components*** of $\overrightarrow{OM}$, and $x\vec{i}$, $y\vec{j}$ and $z\vec{k}$ are called the ***vector components*** of $\overrightarrow{OM}$ along the respective axes. Sometimes x, y and z are also termed as ***rectangular components***.

The ***magnitude*** (or ***length***) of any vector $\vec{a}=(x,y,z)$ is given by

$$|\vec{a}|=\sqrt{x^2+y^2+z^2}\ .$$

If $A(x_1;y_1;z_1)$ and $B(x_2;y_2;z_2)$ are any two points, then the vector joining A and B is the vector $\overrightarrow{AB}$. The components of $\overrightarrow{AB}$ can be determined as

$$\overrightarrow{AB}=(x_2-x_1)\vec{i}+(y_2-y_1)\vec{j}+(z_2-z_1)\vec{k}$$

and its magnitude is given by

$$\left|\overrightarrow{AB}\right|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\ .$$

If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form $\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$, $\vec{b}=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$, then

1) the sum (subtraction) of the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a}\pm\vec{b}=(a_x\pm b_x)\vec{i}+(a_y\pm b_y)\vec{j}+(a_z\pm b_z)\vec{k}\ ;$$

2) the multiplication of a vector $\vec{a}$ by any scalar λ is given by

$$\lambda\vec{a}=\lambda a_x\vec{i}+\lambda a_y\vec{j}+\lambda a_z\vec{k}\ ;$$

3) the vectors $\vec{a}$ and $\vec{b}$ are equal if and only if

$$a_x=b_x,\ a_y=b_y,\ a_z=b_z\ .$$

Example 4.1 Four points $A(5,1)$, $B(6,-2)$, $C(-6,-8)$, $D(-4,-4)$ are given on the coordinate plane. It is known that $\vec{c}=-2\overrightarrow{AB}+3\overrightarrow{CD}$, $\vec{d}=2\overrightarrow{AB}-\frac{1}{2}\overrightarrow{CD}$. Write vectors $\vec{c}$ and $\vec{d}$ in coordinate form and find the magnitudes of these vectors.

Solution. Let us write down the vectors $\vec{c}$ and $\vec{d}$ in the coordinate form:

$$\overrightarrow{AB}=(6-5,-2-1)=(1,-3),\quad \overrightarrow{CD}=(-4-(-6),-4-(-8))=(2,4),$$

$$-2\overrightarrow{AB}=-2\cdot(1,-3)=(-2\cdot 1,-2\cdot(-3))=(-2,6),$$

$$3\overrightarrow{CD}=3\cdot(2,4)=(3\cdot 2,3\cdot 4)=(6,12),$$

$$2\overrightarrow{AB}=2\cdot(1,-3)=(2\cdot 1,2\cdot(-3))=(2,-6),$$

$$\frac{1}{2}\overrightarrow{CD}=\frac{1}{2}\cdot(2,4)=\left(\frac{1}{2}\cdot 2,\frac{1}{2}\cdot 4\right)=(1,2),$$

$$\vec{c}=-2\overrightarrow{AB}+3\overrightarrow{CD}=(-2,6)+(6,12)=(-2+6,6+12)=(4,18),$$

$$\vec{d}=2\overrightarrow{AB}-\frac{1}{2}\overrightarrow{CD}=(2,-6)-(1,2)=(2-1,-6-2)=(1,-8).$$

Find the magnitudes of vectors $\vec{c}$ and $\vec{d}$:

$$\left|\vec{c}\right|=\sqrt{4^2+18^2}=\sqrt{16+324}=\sqrt{340}=2\sqrt{85},$$

$$\left|\vec{d}\right|=\sqrt{1^2+(-8)^2}=\sqrt{1+64}=\sqrt{65}.$$

Remarks.

1. One may observe that whatever be the value of λ, the vector $\lambda\vec{a}$ is always collinear to the vector $\vec{a}$. In fact, two vectors $\vec{a}$ and $\vec{b}$ are ***collinear*** if and only if there exists a nonzero scalar λ such that $\vec{b}=\lambda\vec{a}$. If the vectors $\vec{a}$ and $\vec{b}$ are given in the component form, then the two vectors are collinear if and only if

$$\frac{a_x}{b_x}=\frac{a_y}{b_y}=\frac{a_z}{b_z}=\lambda.$$

2. The angles formed by a vector $\vec{a}$ with coordinate axes Ox,

Oy and Oz are determined from the formulas:

$$\cos\alpha = \frac{a_x}{|\vec{a}|} = \frac{a_x}{\sqrt{a_x^2 + a_y^2 + a_z^2}}, \quad \cos\beta = \frac{a_y}{|\vec{a}|} = \frac{a_y}{\sqrt{a_x^2 + a_y^2 + a_z^2}},$$

$$\cos\gamma = \frac{a_z}{|\vec{a}|} = \frac{a_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}}.$$

The cosines defined by these formulas are called the ***direction cosines*** of the vector. The sum of the squares of all the direction cosines of a vector is equal to one:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

3. The ***projection*** of a vector $\vec{a}$ onto a nonzero vector $\vec{b}$, $\vec{b} \neq 0$, is a number, which is denoted as $pr_{\vec{b}}\vec{a}$ and calculated by the formula

$$pr_{\vec{b}}\vec{a} = |\vec{a}|\cos\varphi,$$

where φ is the angle between vectors $\vec{a}$ and $\vec{b}$, $0 \leq \varphi \leq \pi$ (Figure 4.3).

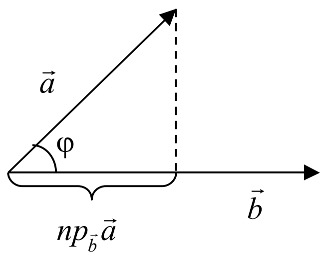


Figure 4.3

Example 4.2 Set whether vectors $\vec{b} = 2\vec{i} - 6\vec{j} + 4\vec{k}$ and $\vec{c} = \vec{i} - 3\vec{j} + 2\vec{k}$ are collinear?

Solution. Thus, the proportion of the given vectors is equal to the same nonzero scalar

$$\frac{2}{1} = \frac{-6}{-3} = \frac{4}{2} = 2,$$

therefore, vectors $\vec{b}$ and $\vec{c}$ are collinear.

Example 4.3 Find the direction cosines of the vector $\vec{a} = \overline{AB}$, if $A(-2,1,0)$, $B(1,2,-1)$.

Solution. Find the components and length of the vector $\overrightarrow{AB}$:

$$\vec{a} = (1-(-2))\cdot\vec{i} + (2-1)\cdot\vec{j} + (-1-0)\cdot\vec{k} = 3\vec{i} + \vec{j} - \vec{k},$$

$$|\vec{a}| = \sqrt{3^2 + 1^2 + (-1)^2} = \sqrt{9+1+1} = \sqrt{11}.$$

Find the direction cosines of the vector $\overrightarrow{AB}$:

$$\cos\alpha = \frac{a_x}{|\vec{a}|} = \frac{3}{\sqrt{11}}, \quad \cos\beta = \frac{a_y}{|\vec{a}|} = \frac{1}{\sqrt{11}}, \quad \cos\gamma = \frac{a_z}{|\vec{a}|} = -\frac{1}{\sqrt{11}}.$$

The ***scalar product*** of two nonzero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}\cdot\vec{b}$, is defined as

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\varphi,$$

where φ is the angle between $\vec{a}$ and $\vec{b}$.

Properties of the scalar product:

1) the scalar product is commutative, i.e.

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a};$$

2) let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then

$$(\lambda\vec{a})\cdot\vec{b} = \vec{a}\cdot(\lambda\vec{b}) = \lambda\vec{a}\cdot\vec{b};$$

3) distributivity of a scalar product over addition.

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c};$$

4) $$(\vec{a})^2 = \vec{a}\cdot\vec{a} = |\vec{a}|\cdot|\vec{a}|\cdot\cos 0 = |\vec{a}|^2.$$

Directly from the definition we get that the angle between two nonzero vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\varphi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}.$$

Projection of a vector $\vec{a}$ on other vector $\vec{b}$ is given by

$$pr_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}.$$

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are ***orthogonal*** (or ***perpendicular***) to each other if and only if their scalar product is equal to zero:

$$\vec{a}\perp\vec{b} \Leftrightarrow \vec{a}\cdot\vec{b} = 0.$$

For mutually perpendicular unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$, we have

$$\left(\vec{i}\right)^2 = \left(\vec{j}\right)^2 = \left(\vec{k}\right)^2 = 1, \quad \vec{i}\cdot\vec{j} = \vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{k} = 0.$$

If two vectors $\vec{a}$ and $\vec{b}$ are given in component form as $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$, $\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$, then their scalar product is given as

$$\vec{a}\cdot\vec{b} = \left(a_x\vec{i} + a_y\vec{j} + a_z\vec{k}\right)\left(b_x\vec{i} + b_y\vec{j} + b_z\vec{k}\right) =$$

$$= a_xb_x\left(\vec{i}\right)^2 + a_xb_y\vec{i}\cdot\vec{j} + a_xb_z\vec{i}\cdot\vec{k} + a_yb_x\vec{j}\cdot\vec{i} + a_yb_y\left(\vec{j}\right)^2 + a_yb_z\vec{j}\cdot\vec{k} + a_zb_x\vec{k}\cdot\vec{i} +$$

$$+ a_zb_y\vec{k}\cdot\vec{j} + a_zb_z\left(\vec{k}\right)^2 = a_xb_x + a_yb_y + a_zb_z.$$

Example 4.4 Find the cosine of the angle BAC and the projection of side AB onto the side AC, if $A(-4;-2;0)$, $B(-1;-2;4)$, $C(3;-2;1)$ are the vertices of the triangle.

Solution. Find the cosine of the angle BAC as the cosine of the angle between two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\cos(\overrightarrow{AB},\ \hat{}\ \overrightarrow{AC}) = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right| \cdot \left|\overrightarrow{AC}\right|},$$

$$\overrightarrow{AB} = \left(-1-(-4);-2-(-2);4-0\right) = (3;0;4),$$

$$\overrightarrow{AC} = \left(3-(-4);-2-(-2);1-0\right) = (7;0;1),$$

$$\left|\overrightarrow{AB}\right| = \sqrt{3^2+0^2+4^2} = \sqrt{25} = 5,$$

$$\left|\overrightarrow{AC}\right| = \sqrt{7^2+0^2+1^2} = \sqrt{50} = 5\sqrt{2},$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = 3 \cdot 7 + 0 \cdot 0 + 4 \cdot 1 = 21 + 4 = 25,$$

$$\cos(\overrightarrow{AB},\ \hat{}\ \overrightarrow{AC}) = \frac{25}{5 \cdot 5\sqrt{2}} = \frac{25}{25\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Find the projection of side AB to side AC by the formula:

$$pr_{\overrightarrow{AC}}\overrightarrow{AB} = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AC}\right|} = \frac{25}{5\sqrt{2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}.$$

Example 4.5 Prove that the vectors $\vec{p} = \vec{a} - \frac{\vec{b}\left(\vec{b}\vec{a}\right)}{\vec{b}^2}$ and $\vec{b}$ are orthogonal.

Solution. As we know, vectors are orthogonal if their scalar product is equal to zero. Check this, multiply the vectors

$$\vec{b}\vec{p} = \vec{b}\left(\vec{a} - \frac{\vec{b}\left(\vec{b}\vec{a}\right)}{\vec{b}^2}\right) = \vec{b}\vec{a} - \frac{\vec{b}\vec{b}\left(\vec{b}\vec{a}\right)}{\vec{b}^2} = \vec{b}\vec{a} - \frac{\vec{b}^2\left(\vec{b}\vec{a}\right)}{\vec{b}^2} = \vec{b}\vec{a} - \vec{b}\vec{a} = 0.$$

Since $\vec{b}\vec{p} = 0$ then $\vec{b}$ and $\vec{p}$ are orthogonal vectors ($\vec{b} \perp \vec{p}$).

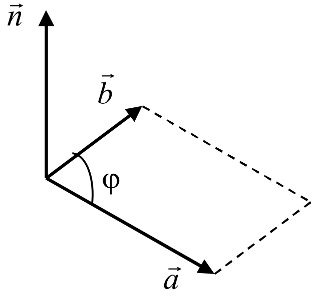


Figure 4.4

The ***vector product*** of two nonzero vectors $\vec{a}$ and $\vec{b}$, is denoted by $\vec{a}\times\vec{b}$ and defined as

$$\vec{a}\times\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\sin\varphi\vec{n},$$

where φ is the angle between $\vec{a}$ and $\vec{b}$ ($0\le\varphi\le\pi$), $\vec{n}$ is a unit vector that is perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}$, $\vec{b}$ and $\vec{n}$ form a right handed system (Figure 4.4), i.e., the right handed system rotated from $\vec{a}$ to $\vec{b}$ moves in the direction of $\vec{n}$.

Properties of the vector product:

1) the vector product is not commutative:

$$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a};$$

2) let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then

$$\left(\lambda\vec{a}\right)\times\vec{b}=\vec{a}\times\left(\lambda\vec{b}\right)=\lambda\vec{a}\times\vec{b};$$

3) let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then

$$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c},$$

it is a distributivity property of a scalar product over addition;

4) let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Then $\vec{a}\times\vec{b}=0$ if and only if $\vec{a}$ and $\vec{b}$ are parallel (or collinear) to each other, i.e.,

$$\vec{a}\times\vec{b}\Leftrightarrow\vec{a}\perp\vec{b}.$$

In particular, $\vec{a}\times\vec{a}=0$.

For mutually perpendicular unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$, we have

$$\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0,$$

$$\vec{i}\times\vec{j}=\vec{k}\,,\ \ \vec{j}\times\vec{k}=\vec{i}\,,\ \vec{k}\times\vec{i}=\vec{j}\,,$$

$$\vec{j}\times\vec{i}=-\vec{k}\,,\ \ \vec{k}\times\vec{j}=-\vec{i}\,,\ \vec{i}\times\vec{k}=-\vec{j}\,.$$

Let $\vec{a}$ and $\vec{b}$ be two vectors given in component form as $\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$, $\vec{b}=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$. Then their cross product may be given by

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}.$$

If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram then its area is given by $S_\Delta=\left|\vec{a}\times\vec{b}\right|$.

If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle then its area is given as

$$S_\Delta=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|.$$

Example 4.6 Find the area of a triangle having the points $A(2;2;1)$, $B(3;0;3)$, $C(13;4;11)$ as its vertices.

Solution:

$$\overrightarrow{AB}=(3-2;0-2;3-1)=(1;-2;2),$$

$$\overrightarrow{AC}=(13-2;4-2;11-1)=(11;2;10),$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 2\\ 11 & 2 & 10\end{vmatrix}=\vec{i}\begin{vmatrix}-2 & 2\\ 2 & 10\end{vmatrix}-\vec{j}\begin{vmatrix}1 & 2\\ 11 & 10\end{vmatrix}+\vec{k}\begin{vmatrix}1 & -2\\ 11 & 2\end{vmatrix}=$$

$$=\vec{i}(-20-4)-\vec{j}(10-22)+\vec{k}(2+22)=-24\vec{i}+12\vec{j}+24\vec{k}\,,$$

$$\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \sqrt{(-24)^2+12^2+24^2} = \sqrt{1296} = 36 .$$

Thus, the required area is $S_{ABC} = \frac{1}{2}\cdot 36 = 18$.

The ***scalar triple product*** is the scalar product of a vector product and a third vector, i.e. $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$. It can be represented as the determinant

$$\vec{a}\,\vec{b}\,\vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

The scalar triple product gives the volume of the parallelepiped whose sides are represented by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$:

$$V = \left|\vec{a}\,\vec{b}\,\vec{c}\right|.$$

The volume of a triangular pyramid built on vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is equal to

$$V_{\text{pyramid}} = \frac{1}{6}\left|\vec{a}\,\vec{b}\,\vec{c}\right|.$$

Example 4.7 Find the volume of the pyramid having the points $A(1,-4,0)$, $B(5,0,-2)$, $C(3,7,-10)$, $D(1,-2,1)$ as its vertices.

Solution: find the coordinates of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ on which the pyramid is built:

$$\vec{a} = \overrightarrow{AB} = \left(5-1, 0-(-4), -2-0\right) = (4,4,-2),$$

$$\vec{b} = \overrightarrow{AC} = \left(3-1, 7-(-4), -10-0\right) = (2,11,-10),$$

$$\vec{c} = \overrightarrow{AD} = \left(1-1, -2-(-4), 1-0\right) = (0,2,1).$$

Find the scalar triple product:

$$\vec{a}\vec{b}\vec{c}=\begin{vmatrix}4 & 4 & -2\\ 2 & 11 & -10\\ 0 & 2 & 1\end{vmatrix}=44-8+0-0-8+80=108.$$

Thus, the required volume is $V=\dfrac{1}{6}\cdot 108=18$.

The parallelepiped would have zero volume. In this case all three vectors lie in the same plane (they are coplanar). Thus, three vectors are ***coplanar***, if and only if scalar triple product of them is equal to zero:

$$\vec{a}\vec{b}\vec{c}=0.$$

If three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are not coplanar, then they form a basis, i.e. any vector $\vec{d}$ can be submitted as

$$\vec{d}=x\vec{a}+y\vec{b}+z\vec{c}.$$

This equality is called the ***decomposition of the vector*** $\vec{d}$ ***in the basis*** $\{\vec{a},\vec{b},\vec{c}\}$. Numbers x, y, z are components of the vector $\vec{d}$ in this basis.

If the components of the basis vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and the vector $\vec{d}$ in the coordinate basis $\{\vec{i},\vec{j},\vec{k}\}$ are known, then, writing the decomposition of the vector $\vec{d}$ over the new basis $\{\vec{a},\vec{b},\vec{c}\}$ in a scalar form, we obtain a system of linear equations

$$\begin{cases} a_x x+b_x y+c_x z=d_x;\\ a_y x+b_y y+c_y z=d_y;\\ a_z x+b_z y+c_z z=d_z \end{cases}$$

for finding the new components x, y, z of the vector $\vec{d}$.

Example 4.8 Show that the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a basis and find the decomposition of the vector $\vec{d}$ in this basis.

$$\vec{a}=(2;-1;4),\ \vec{b}=(1;-2;2),\ \vec{c}=(-1;2;1),\ \vec{d}=(-4;14;7).$$

Solution. Calculate the scalar triple product of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$:

$$\vec{a}\vec{b}\vec{c}=\begin{vmatrix} 2 & -1 & 4 \\ 1 & -2 & 2 \\ -1 & 2 & 1 \end{vmatrix}=-4+8+2-8+1-8=-9\neq 0,$$

therefore, vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are not coplanar and they form a basis. Find the decomposition of the vector $\vec{d}$ in this basis.

Construct a system of equations and solve it by the Cramer's rule:

$$\begin{cases} 2x+y-z=-4; \\ -x-2y+2z=14; \\ 4x+2y+z=7, \end{cases} \quad \Delta=\begin{vmatrix} 2 & 1 & -1 \\ -1 & -2 & 2 \\ 4 & 2 & 1 \end{vmatrix}=-9,$$

$$\Delta_1=\begin{vmatrix} -4 & 1 & -1 \\ 14 & -2 & 2 \\ 7 & 2 & 1 \end{vmatrix}=8-28+14-14-14+16=-18,$$

$$\Delta_2=\begin{vmatrix} 2 & -4 & -1 \\ -1 & 14 & 2 \\ 4 & 7 & 1 \end{vmatrix}=28+7-32+56-4-28=27,$$

$$\Delta_3=\begin{vmatrix} 2 & 1 & -4 \\ -1 & -2 & 14 \\ 4 & 2 & 7 \end{vmatrix}=-28+8+56-32+7-56=-45.$$

$$x = \frac{-18}{-9} = 2, \quad y = \frac{27}{-9} = -3, \quad z = \frac{-45}{-9} = 5.$$

Let us check the obtained solution:

$$\begin{cases} 2 \cdot 2 + (-3) - 5 = 4 - 3 - 5 = -4; \\ -2 - 2 \cdot (-3) + 2 \cdot 5 = -2 + 6 + 10 = 14; \\ 4 \cdot 2 + 2 \cdot (-3) + 5 = 8 - 6 + 5 = 7. \end{cases}$$

Thus, the decomposition of the vector $\vec{d}$ in the basis $\{\vec{a}, \vec{b}, \vec{c}\}$ is

$$\vec{d} = 2\vec{a} - 3\vec{b} + 5\vec{c}.$$

Example 4.9 Prove that the vectors $\vec{p} = \vec{a} - \frac{\vec{b}(\vec{b}\vec{a})}{\vec{b}^2}$ and $\vec{b}$ are orthogonal.

Solution. As we know, vectors are orthogonal if their scalar product is equal to zero. Check this, multiply the vectors

$$\vec{b}\vec{p} = \vec{b}\left(\vec{a} - \frac{\vec{b}(\vec{b}\vec{a})}{\vec{b}^2}\right) = \vec{b}\vec{a} - \frac{\vec{b}\vec{b}(\vec{b}\vec{a})}{\vec{b}^2} = \vec{b}\vec{a} - \frac{\vec{b}^2(\vec{b}\vec{a})}{\vec{b}^2} = \vec{b}\vec{a} - \vec{b}\vec{a} = 0.$$

Since $\vec{b}\vec{p} = 0$ then $\vec{b}$ and $\vec{p}$ are orthogonal vectors ($\vec{b} \perp \vec{p}$).

Lecture 5
Basic concepts of tensor calculus

Tensor calculus is a multidimensional generalization of matrix algebra. It is very useful in the analysis of multidimensional linear systems, such as the wideband MIMO channel. In particular, the HOSVD can be used to decompose a higher order tensor into several orthogonal bases, one for each dimension of the tensor, and a core tensor that describes the interaction between the bases. The *n*-th orthogonal basis is computed by computing the SVD of the tensor's *n*-th unfolding. The HOSVD, and the relevant tensor algebra, provide the inspiration for the structured model.

A *tensor* is an object carrying upper (contravariant) and lower (covariant) indices, each running through D different values, and transforming when the coordinates of a certain class are replaced in a certain linear way, in which the zero tensor (all components of which are equal to zero) remains zero in any coordinates. Here D is a dimension of a space. For a 4-dimensional tensor in a 4-dimensional space-time we have:

$$T^{ij\ldots}{}_{mn\ldots},\ \ i, j, \ldots m, n \ldots \in \{0,1,2,3\}.$$

For a 3-dimensional tensor in a 3-dimensional space-time is:

$$T^{\alpha\beta\ldots}{}_{\mu\nu\ldots},\ \ \alpha, \beta, \ldots \mu, \nu \ldots \in \{1,2,3\}.$$

Hereinafter, small Latin indices will run from 0 to 3, and Greek indices will run from 1 to 3. Tensor valence or tensor rank is the total number of indices. Do not confuse tensor rank (number of indices) and matrix rank (number of linearly independent columns / rows). These are different concepts. If a tensor has two indices, then it has the rank (valence) of the tensor 2, and the rank of the corresponding matrix can be any integer from zero to the dimension of the space. The tensor generalizes the concepts of scalar, vector and matrix. Moreover, the transformation rules for the tensor components are arranged so that we can construct new tensors from the existing ones according to some simple rules. Other objects that carry indices can also be used, but they are transformed according to different rules and are not tensors.

Note that vectors in different areas of mathematics call different objects. As a rule, an element of linear space is called a vector, i.e. vectors can be multiplied by a number and added. Our vectors will also allow these operations, i.e. will be elements of some linear space. Elements of a certain linear space will be covectors, as well as any tensors of a certain type (with a certain number of superscripts and subscripts). However, the word "vector" will further mean not only belonging to a linear space, but also a certain transformation law. Immediately we will stipulate that the coordinates will carry the superscript ($i, j, \ldots$, or $\alpha, \beta, \ldots$), but the set of coordinates ("radius vector") can be considered a vector only if we restrict ourselves to linear transformations that leave the origin of coordinates fixed.

A *scalar* is a tensor without indices. It has one component. When changing coordinates, the scalar is not transformed, i.e. is an *invariant*.

One and the same vector can be written in both covariant and contravariant components. Usually, some of them are natural for the under consideration vector. The coordinates of the geometric vector (displacement vector) are naturally contravariant. A contravariant vector is denoted in the form: x^i, that is, with a index at the top. The components of the covariant vector change, as it were, opposite to the change in the basis vectors (hence its name). For example, Let us pass from one coordinate system to another one, such that:

$$x'^1 = Nx^1,\ x'^2 = x^2,\ x'^3 = x^3\ \ (N > 1).$$

In other words, we changed the scale of the first axis, making it smaller. The new unit of length on this axis has decreased, and is $\frac{1}{N}$ from the old. And the corresponding new coordinate of the vector, on the contrary, has increased in N times, as if opposite to the scale of the axis. This is contravariance. For a covariant vector, the opposite is true ... Although the same displacement vector can be represented in covariant form: x_i . Its covariant components: x_1, x_2, x_3 are components not in the basis of our task. And in some other (dual) coordinate system. We just know how to go to it: through the coefficients g_{ik}. And so we keep such a transition in a mind.

Einstein's rule is that by an index that occurs twice (once at the top, another time at the bottom) summation is meant. So, $a_i b^i$ is a shorthand expression of $\sum_i a_i b^i$. Here the index i (running through the values 1, 2, 3) is called *dumb*: it is not included in the resulting expression it seems to be “reduced”. In such cases, it is said that the convolution has been performed.

Let us rewrite again formulas (5.1) for obtaining covariant components from the original contravariant ones:

$$\begin{aligned} x_1 &= g_{11}x^1 + g_{12}x^2 + g_{13}x^3, \\ x_2 &= g_{21}x^1 + g_{22}x^2 + g_{23}x^3, \\ x_{13} &= g_{31}x^1 + g_{32}x^2 + g_{33}x^3. \end{aligned} \tag{5.1}$$

Look at the each expression from (5.1). We will see in them the result of multiplying two matrices (5.1a):

$$[x_1, x_2, x_3] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{23} & g_{33} \end{bmatrix} \times \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}. \tag{5.1a}$$

Remember we mentioned that a vector is a special case of a tensor? It’s time to clarify that the vector is a tensor of the first rank (sometimes instead of “rank” they say “valence”). A contravariant vector is usually represented by a column matrix. Covariant vector is usually represented by a row matrix. We are introduced here to the second-rank tensor represented by the matrix 3×3 (g_{ik}). A tensor of the third rank will have to be imagined as a three-dimensional table. The number of indices in a symbolic notation corresponds to the rank of the tensor. And the sizes of rows and columns always correspond to the number of dimensions of space (in our example, it is three-dimensional).

Let us write (5.1) in shorthand:

$$x_i = g_{ik} x^k . \tag{5.2}$$

What can we see from (5.2)? So much.

1. This is an entry shortened by Einstein's rule. In full, it looks like this:

$$x_i = \sum_i g_{ik} x^k . \tag{5.2a}$$

Here k is a dumb index (not included in the result).

2. g_{ik} is a symbol for the second rank covariant tensor, because there is indexes below. And it is at the bottom, because there is a rule: repeating indices must alternate (top-bottom). When combining covariant components and contravariant components, the laws of its transformation are they are mutually simplified. Otherwise, the final result will not be a tensor – it will lose invariance!

3. The result is a covariant vector (i at the bottom), because the right index i is covariant.

4) The number of dimensions of space (the number of values that the summation index runs through) is not clearly visible here, and should be understood from the context of the task. Accordingly, by (5.1a) one should actually mean three formulas: for $i = 1,2,3$.

For tensors are possible to add component wise tensors of the same structure, multiply them by a number - we will not dwell on this too much. The space of tensors, as in the case of vectors, is assumed to be linear, that is, the result of such operations will again be a tensor. By and large, you have to keep in mind two basic operations with tensors: multiplication and convolution. These operations with tensors lead to the same tensors. Here is an illustration of the tensor product:

$$x_i x_k = X^{ik} = \begin{bmatrix} x^1x^1 & x^2x^1 & x^3x^1 \\ x^1x^2 & x^2x^2 & x^3x^2 \\ x^1x^3 & x^2x^3 & x^3x^3 \end{bmatrix}.$$

As you can see, the resulting tensor X^{ik} it is a tensor of a total rank. It contains components equal to the product of the components of the factors - each with each. And all the indices of the factors simply went over to the product. Important: the fact that the product of two vectors is a tensor of the 2nd rank does not at all follow that any such tensor can be represented as a product of some vectors! For example, the product: $A^{ik}B_{lmn} = C^{ik}_{lmn}$ will be the tensor of the 5-th rank. Moreover, as they say, it is mixed: it is twice contravariant and three times covariant.

We have already noted, using the example of the square of the length of a vector, that the simplest convolution is the dot product. Let's consider the question in more details. Convolution appears during the multiplication record, when one of the indices is repeated above and below. So, the product: $A^{ik}B_{lmn} = C^{ik}_{mn}$ will not have 5-th rank, it will have the third rank: during a single convolution, the rank decreases by 2. Here the convolution goes by index i. Remembering that the repeated index means summation, we write our convolution in details:

$$A^{ik}B_{imn} = \sum_i A^{ik}B_{imn} = A^{1k}B_{1mn} + A^{2k}B_{2mn} + A^{3k}B_{3mn}.$$

Here you can see how the dumb index i disappears. The convolution of a tensor of the 2nd rank within itself is called a tensor trace; experts of the old school prefer the German equivalent: *spur*. So the square of the length $x^2 = x_i x^i$ is the tract (spur) of the tensor $X^i_i = x_i x^i$. In fact, this is the sum of the elements of its main diagonal. Obviously, the trace, like any scalar, is an invariant.

We know that the invariant of a vector is its length, which is expressed through the convolution of the vector with the co-vector (that is, the conjugate vector). And what about the tensor of higher rank? Any tensor has an invariant (scalar) obtained by convolution with the conjugate tensor. For example, the expression $A^{ik}A_{ik}$ will be an invariant of a tensor of the second rank, twice contravariant: A^{ik}. Let us describe what double convolution is. First, we collapse, for example, by index i:

$$A^{ik}A_{ik} = A^{1k}A_{1k} + A^{2k}A_{2k} + A^{3k}A_{3k}.$$

At the second step, we fold each of the three resulting terms by k:

$$A^{ik}A_{ik} = \left(A^{11}A_{11} + A^{12}A_{12} + A^{13}A_{13}\right) + \left(A^{21}A_{21} + A^{22}A_{22} + A^{23}A_{23}\right) + \\ + \left(A^{31}A_{31} + A^{32}A_{32} + A^{33}A_{33}\right).$$

The resulting expression is invariant, because as a result of double convolution all indices disappear and we get a scalar. Simple rules of index manipulation save us from time consuming proofs. Of course, the invariant of the combined tensor A_i^k will be a value $A_i^k A_k^i$.

Writing the formula for the length of a vector using the dot product as:

$$x^2 = x_1x^1 + x_2x^2 + x_3x^3. \tag{5.3}$$

Now we can write it in a short form:

$$x^2 = x_ix^i. \tag{5.3a}$$

Recall formula (5.2) for x_i and substitute it in (5.3a), we get:

$$x^2 = g_{ik}x_ix^k. \tag{5.4}$$

This is a general form of a tensor expression for a length using contravariant (natural) components. Tensor g_{ik} is a metric tensor of a space. *The metric tensor* is, as it was, a rule for calculating of the length of any vector from the values of its components. In relation to a formula (5.2), they say that here the vector x^i is convolved with the metric tensor and the vector is obtained x_i. That is, the metric tensor is also a way to transform components - from contravariant to covariant and vice versa. Now we have some options (to choose from) for the calculating of a vector length:

$$x^2 = x_i x^i \text{ or } x^2 = g_{ik} x^i x^k \text{ or } x^2 = g_{ik} x_i x_k .$$

Similarly, we have for the dot product:

$$xy = x_i y^i \text{ or } xy = g_{ik} x^i y^k \text{ or } xy = g_{ik} x_i y_k .$$

In the course of the above transformations, we use some properties. First, remember that we introduced the coefficients in the expression for the length (for reasons of symmetry) so, that is: $g_{ik} = g_{ki}$. Here is the first property of the metric tensor: its matrix is symmetric (the elements symmetric about the main diagonal are the same). Further, in Cartesian coordinates we have

$$x^2 = \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2,$$

that is, there is no difference between x^i and x_i. We came to the second conclusion: this is where the metric tensor looks extremely simple (5.5):

$$g_{ik} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.5}$$

Members with mixed indices ($i \neq k$) in the expression of the length are not included. So, the metric tensor for the case of rectangular coordinates is a diagonal matrix, and all elements of the main diagonal are equal to one.

The unit tensor, denoted by the Kronecker symbol δ_i^k is a tensor that is determined as:

$$\delta_i^k x^i = x^k \tag{5.6}$$

for arbitrary vector x. The unit tensor, as it were, highlights the desired vector k-component. The sum is written on the left, let's open it:

$$\delta_i^k x^i = \delta_1^k x^1 + \delta_2^k x^2 + \delta_3^k x^3 .$$

The equality will be fulfilled, if and only if one of components δ_i^k, which has equal indices $i = k$, be equal to one. And the rest should be zero. Hence, the unit tensor looks exactly like (5.5)! Let's go to another coordinate system, the components of the vector will change. And the unit tensor too … But what we have explained regarding (5.6) remains, nevertheless, in force.! It turns out that the tensor δ_i^k has a rare property: its components are the same in any coordinate system, do not change.

Lecture 6
Elements of analytic geometry on a plane

Two mutually perpendicular coordinate lines Ox and Oy with common origin O form a ***Cartesian rectangular coordinate system on a plane***. Ox is called the ***abscissa axis*** (x-axis), and Oy is the ***ordinate axis*** (y-axis). The set of straight lines, perpendicular to the coordinate axes, forms a ***coordinate grid*** on the coordinate plane Oxy. The location of a point is specified by a pair of numbers called the x- and y***-coordinates of a point*** and is written as (x, y).

The ***distance*** d between points $M_1(x_1; y_1)$ and $M_2(x_2; y_2)$ on the plane is determined by the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

The ***coordinates of the point*** $M(x; y)$***, which divides the segment*** M_1M_2 ***in a given ratio*** $\lambda = \dfrac{|M_1M|}{|MM_2|}$, are determined by the formulas:

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda}.$$

If the point M divides the segment M_1M_2 into two equal parts, then $\lambda = 1$. The ***coordinates of the midpoint of the segment*** are determined by the formulas:

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}.$$

The ***area of a triangle*** with vertices at the points $A(x_1; y_1)$, $B(x_2; y_2)$ and $C(x_3; y_3)$ can be calculated by the following formula:

$$S=\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}=\frac{1}{2}\left|x_1y_2+x_2y_3+x_3y_1-x_2y_1-x_3y_2-x_1y_3\right|.$$

Example 6.1 Find the length of the median AK and the area of the triangle ABC with vertices $A(3,4)$, $B(-3,-1)$, $C(3,-2)$.

Solution. As we know that median AK divides the opposite side BC into two equal parts, i.e. point K is the midpoint of the side BC. Then, the coordinates of point K are

$$x=\frac{-3+3}{2}=0,\ y=\frac{-1-2}{2}=-\frac{3}{2}.$$

Find the length of the median AK as the distance between the points A and K:

$$|AK|=\sqrt{(0-3)^2+\left(-\frac{3}{2}-4\right)^2}=\sqrt{(-3)^2+\left(-\frac{11}{2}\right)^2}=\sqrt{9+\frac{121}{4}}=\sqrt{\frac{157}{4}}=\frac{\sqrt{157}}{2}.$$

Calculate the area of the triangle ABC:

$$S=\frac{1}{2}\begin{vmatrix} 3 & 4 \\ -3 & -1 \\ 3 & -2 \\ 3 & 4 \end{vmatrix}=\frac{1}{2}\cdot\left|-3+6+12+12+3+6\right|=\frac{1}{2}\cdot 36=18.$$

Any equation of the first power with respect to x and y, i.e. an equation of the form

$$Ax+By+C=0$$

(where A, B and C are the constant coefficients, with $A^2+B^2\neq 0$) defines a straight line on the plane. This equation is called the ***general equation of a line***.

Special cases of the *general equation of a straight line:*

1) $C = 0$, $A \neq 0$, $B \neq 0$, i.e. $Ax + By = 0$ is the equation of straight line passing through the origin;

2) $A = 0$, $B \neq 0$, $C \neq 0$, i.e. $By + C = 0$ (or $y = b$, where $b = -C / B$) is the equation of straight line parallel to the x-axis;

3) $B = 0$, $A \neq 0$, $C \neq 0$, i.e. $Ax + C = 0$ (or $x = a$, where $a = -C / A$) is the equation of straight line parallel to the y-axis;

4) $B = C = 0$, $A \neq 0$, i.e. $Ax = 0$ (or $x = 0$) is the equation of y-axis;

5) $A = C = 0$, $B \neq 0$, i.e. $By = 0$ (or $y = 0$) is the equation of x-axis.

If in the general equation of the line $B \neq 0$, then, resolving it with respect to y, we obtain an ***equation in the slope-intercept form***:

$$y = kx + b,$$

where $k = tg\alpha = \dfrac{y_2 - y_1}{x_2 - x_1}$ is called the ***slope*** of the line; the angle α, measured counter-clockwise from the positive direction of the x-axis to a line, is called the ***inclination*** of the line; b is the ***y-intercept*** of a graph of the line (Figure 6.1).

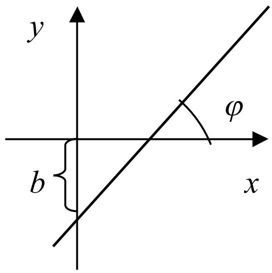


Figure 6.1

The equation of a line passing through two points $M_1(x_1; y_1)$ and $M_2(x_2; y_2)$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

is called the ***two-point form of a line***.

Since $k=\frac{y_2-y_1}{x_2-x_1}$, the two-point form of a line can be reduced to the ***point-slope form***:

$$y-y_1=k\left(x-x_1\right).$$

Example 6.2 Let $A\left(-5;0\right)$, $B\left(3;6\right)$, $C\left(7;-5\right)$ be the vertices of the triangle. Find: 1) the lengths of the sides AB and AC; 2) the equations of the sides AB and AC; 3) the point of the intersection of the medians of a triangle ABC.

Solution.

1. Find the lengths of the sides as the distance between two points:

$$|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{\left(3-(-5)\right)^2+\left(6-0\right)^2}=\sqrt{64+36}=10,$$

$$|AC|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{\left(7-(-5)\right)^2+\left(-5-0\right)^2}=\sqrt{144+25}=13.$$

2. To compose the equations of the sides, we use two-point form of a line.

For AB:

$$\frac{x-(-5)}{3-(-5)}=\frac{y-0}{6-0},\quad \frac{x+5}{8}=\frac{y}{6},\quad 6\left(x+5\right)=8y,\quad 6x+30-8y=0,$$

$3x-4y+15=0$ – general equation of AB,

$y=\frac{3}{4}x+\frac{15}{4}$ – slope-intercept form of the AB equation.

For AC:

$$\frac{x-(-5)}{7-(-5)}=\frac{y-0}{-5-0},\quad \frac{x+5}{12}=\frac{y}{-5},\quad -5\left(x+5\right)=12y,$$

$$-5x-12y-25=0,$$

$5x+12y+25=0$ – general equation of AC,

$y = -\frac{5}{12}x - \frac{25}{12}$ – slope-intercept form of the AC equation.

3. Let us draw two medians: from the vertices A and C (Figure 6.2).

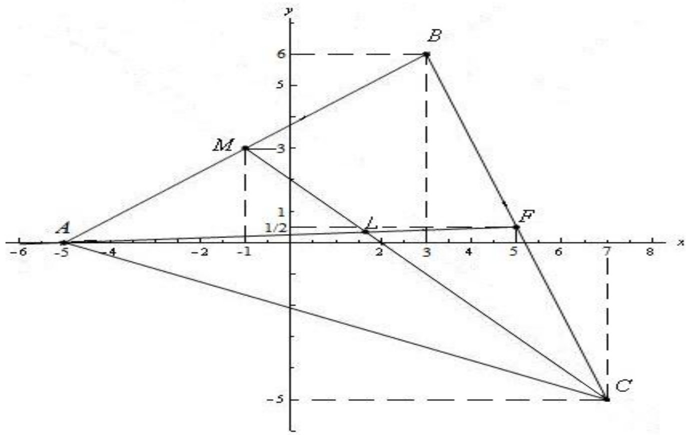


Figure 6.2

First way. The medians of a triangle intersect each other in the ratio 2:1 starting from the vertex.

Find the coordinates of the point L, which divides the median CM in ratio $\lambda = 2$, starting from the vertex C:

$$x = \frac{7 + 2\cdot(-1)}{1+2} = \frac{7-2}{3} = \frac{5}{3}, \quad y = \frac{-5 + 2\cdot 3}{1+2} = \frac{-5+6}{3} = \frac{1}{3}.$$

Thus, $L\left(\frac{5}{3}, \frac{1}{3}\right)$ is the median intersection point.

Second way. Three medians in a triangle intersect at one point and this point is called the center of gravity of the triangle. Its coordinates can be found by formulas:

$$x = \frac{x_1 + x_2 + x_3}{3} = \frac{-5+3+7}{3} = \frac{5}{3},$$

$$y = \frac{y_1 + y_2 + y_3}{3} = \frac{0 + 6 - 5}{3} = \frac{1}{3}.$$

When two straight lines intersect, they form two angles at the point of intersection. One is an acute angle and another is an obtuse one. Both these angles are supplements of each other. By definition, when we say 'angle between two straight lines' we mean the acute angle between two lines and not the obtuse one. However, since the two angles are supplementary, if one is known, we can find the other.

If two lines in the xy-plane are given by the equations in the slope-intercept form $y = k_1 x + b_1$ and $y = k_2 x + b_2$, then the acute angle between them is determined by the formula

$$tg\varphi = \left| \frac{k_2 - k_1}{1 + k_1 k_2} \right|.$$

This formula cannot be used to find the angle between the lines, if one of them is parallel to y-axis, since the slope of the line parallel to y-axis is indeterminate.

For ***parallel lines*** $\varphi = 0$, $tg\varphi = 0$:

$$k_1 = k_2.$$

For ***perpendicular lines*** $\varphi = 90^\circ$, $tg\varphi \to \infty$:

$$k_1 k_2 = -1.$$

Example 6.3 Let $A(4,0)$, $B(7,4)$, $C(8,2)$ (Figure 6.3) be the vertices of the triangle. Find: 1) angle BAC; 2) the equation of the altitude CD; 3) the equation of a line l passing through the midpoint of AC parallel to side AB; 4) the point of intersection of CD and l.

Solution. We should find four answers to the four questions.

1. The angle BAC is acute and formed by two straight lines AB and AC. Find the slopes of these lines:

$$k_{AB}=\frac{4-0}{7-4}=\frac{4}{3},\quad k_{AC}=\frac{2-0}{8-4}=\frac{1}{2},$$

then $$\operatorname{tg}A=\left|\frac{k_{AB}-k_{AC}}{1+k_{AB}\cdot k_{AC}}\right|=\left|\frac{\frac{4}{3}-\frac{1}{2}}{1+\frac{4}{3}\cdot\frac{1}{2}}\right|=\left|\frac{\frac{8-3}{6}}{\frac{6+4}{6}}\right|=\frac{5}{6}\cdot\frac{6}{10}=\frac{1}{2},$$

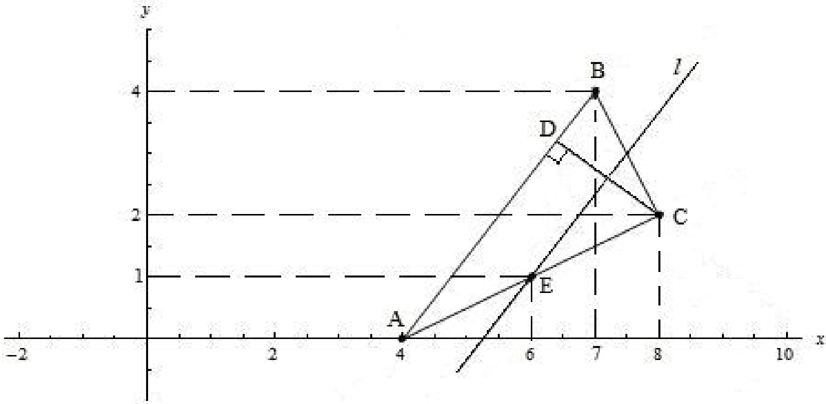


Figure 6.3

$$\angle BAC=\operatorname{arctg}\left(\frac{1}{2}\right).$$

2. The altitude CD is perpendicular to the side AB, so $k_{AB}k_{CD}=-1$ and

$$k_{CD}=-\frac{1}{k_{AB}}=-\frac{1}{\frac{4}{3}}=-\frac{3}{4}.$$

Also it passes through the point $C(8,2)$. According to the point-slope form of straight line equation, we get:

$$y-2=-\frac{3}{4}(x-8),\quad y-2=-\frac{3}{4}x+6,\quad y=-\frac{3}{4}x+8.$$

3. Find the coordinates of the point E that is the midpoint of the side AC:

$$x=\frac{4+8}{2}=\frac{12}{2}=6,\quad y=\frac{0+2}{2}=\frac{2}{2}=1.$$

The straight line l to be determined is parallel to the side AB, so $k_l=k_{AB}=\frac{4}{3}$. Also it passes through the point $E(6,1)$. According to the point-slope form of straight line equation, we get:

$$y-1=\frac{4}{3}(x-6),\quad y-1=\frac{4}{3}x-8,\quad y=\frac{4}{3}x-7.$$

4. To find the coordinates of point P of intersection of CD and l we should solve the system which contains the equations of these lines:

$$\begin{cases} y=-\frac{3}{4}x+8; \\ y=\frac{4}{3}x-7, \end{cases}$$

$$-\frac{3}{4}x+8=\frac{4}{3}x-7,\quad \frac{4}{3}x+\frac{3}{4}x=8+7,\quad \frac{25}{12}x=15,$$

$$x=15\cdot\frac{12}{25}=3\cdot\frac{12}{5}=\frac{36}{5},$$

$$y=\frac{4}{3}\cdot\frac{36}{5}-7=\frac{48}{5}-7=\frac{48-35}{5}=\frac{13}{5}.$$

Thus, $P\left(\frac{36}{5},\frac{13}{5}\right)$.

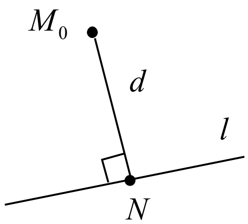


Figure 6.4

Pay attention to the issue how to find the distance between point that are not located on the line and this straight line (Figure 6.4).

If the straight line l is given by its general equation $Ax+By+C=0$, and the coordinates of the point M_0 are x_0 and y_0, then the ***distance from the point*** $M_0(x_0, y_0)$ ***to the line*** is calculated by the formula:

$$d=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}.$$

Example 6.4 Find the length of the altitude CD from the example 6.3.

Solution. Find the equation of the line AB by applying the two-point form of a line:

$$\frac{x-4}{7-4}=\frac{y-0}{4-0}, \quad \frac{x-4}{3}=\frac{y}{4},$$

$$4(x-4)=3y, \quad 4x-16=3y, \quad 4x-3y-16=0.$$

Find the length of the altitude CD as a distance from the point $C(8,2)$ to the line AB:

$$|CD|=\frac{|4\cdot 8-3\cdot 2-16|}{\sqrt{4^2+(-3)^2}}=\frac{|32-6-16|}{\sqrt{16+9}}=\frac{10}{\sqrt{25}}=\frac{10}{5}=2.$$

Example 6.5 Compose the equation of a line passing through a point $K(3,-2)$ and it is perpendicular to the line $2x+5y-7=0$.

Solution. The desired line is perpendicular to the line $2x+5y-7=0$, then we have a fair equality for its slopes of these lines as $k_2 \cdot k_1 = -1$. Find the slope of a given line and, substituting it in this equality, determine the slope of the desired line.

$$2x+5y-7=0,\ 5y=-2x+7,\ y=-\frac{2}{5}x+\frac{7}{5},\ k_1=-\frac{2}{5}.$$

$$k_2=-\frac{1}{k_1}=-\frac{1}{-2/5}=\frac{5}{2}.$$

We know that the slope of the required line, the coordinates of the point belonging to this line, so we make the equation using the point-slope form and get a final answer:

$$y-(-2)=\frac{5}{2}(x-3),\ 2(y+2)=5(x-3),\ -5x+2y+19=0.$$

Example 6.6 Find the coordinates of the lines intersection point if we have equations of these lines $3x+4y-20=0$ and $x-y-2=0$.

Solution. To find the coordinates of lines intersection point we should to solve the system of the given equations

$$\begin{cases}3x+4y-20=0,\\ x-y-2=0;\end{cases} \begin{cases}3x+4y-20=0,\\ x=y+2;\end{cases} \begin{cases}3(y+2)+4y-20=0,\\ x=y+2;\end{cases}$$

$$\begin{cases}6y+6+4y-20=0,\\ x=y+2;\end{cases} \begin{cases}10y-14=0,\\ x=y+2;\end{cases} \begin{cases}y=\frac{7}{5},\\ x=y+2;\end{cases} \begin{cases}y=\frac{7}{5},\\ x=\frac{17}{5}.\end{cases}$$

The lines intersection point has coordinates $x=\frac{17}{5}$ and $y=\frac{7}{5}$.

Lecture 7
PLANE AND LINE IN SPACE

We know that a line is determined by two points. In other words, for any two distinct points, there is exactly one line that passes through those points, whether in two dimensions or three. Similarly, given any three points that do not all lie on the same line, there is a unique plane that passes through these points. Just as a line is determined by two points, a plane is determined by three.

This may be the simplest way to characterize a plane, but we can use other descriptions as well. For example, given two distinct, intersecting lines, there is exactly one plane containing both lines. A plane is also determined by a line and any point that does not lie on the line. These characterizations arise naturally from the idea that a plane is determined by three points. Perhaps the most surprising characterization of a plane is actually the most useful.

Imagine a pair of orthogonal vectors that share an initial point. Visualize grabbing one of the vectors and twisting it. As you twist, the other vector spins around and sweeps out a plane. Here, we describe that concept mathematically. Let $\vec{N} = \{A, B, C\}$ be a vector and $M_0(x_0, y_0, z_0)$ be a point. Then the set of all points $M(x, y, z)$ such that $\overrightarrow{M_0M}$ is orthogonal to $\vec{N}$ forms a plane α (Figure 7.1).

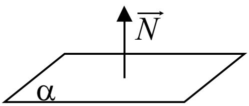


Figure 7.1

We say that $\vec{N}$ is a normal vector, or perpendicular to the plane. Remember, the scalar (dot) product of orthogonal vectors is zero. This fact generates the vector equation of a plane: $\vec{N} \cdot \overrightarrow{M_0M} = 0$. Rewriting this equation provides additional ways to describe the plane:

$$\overrightarrow{M_0M} = (x - x_0, y - y_0, z - z_0),\ \vec{N} \cdot \overrightarrow{M_0M} = 0,$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

A plane in space is the set of all terminal points of vectors emanating from a given point perpendicular to a fixed vector.

If we open parentheses in this equation we get a new form of a plane equation in space as

$$Ax + By + Cz + D = 0,$$

it is a general equation of a plane in space, where coefficients A, B, C can not be equal to zero at the same time. Consider some cases when one or two of coefficients equal zero.

1. If $D = 0$, $\overline{N} = \{A;B;C\}$ and $O(0,0,0) \in \alpha$ then we get the plane equation in a form $Ax + By + Cz = 0$, and it is a plane passing through the origin, that is, the free term D indicates the distance at which the desired plane is located from the origin.

2. If $A = 0, D \neq 0$ then $\overline{N} = \{0;B;C\}$ and $\overline{N} \perp Ox$, and we get an plane equation in a form $By + Cz + D = 0$, it is a plane α that is parallel to the axis Ox. So, if $B = 0, D \neq 0$, $\overline{N} = \{A;0;C\}$, $\overline{N} \perp Oy$ then we get an equation in a form $Ax + Cz + D = 0$ and we have a plane α that is parallel to the axis Oy. If $C = 0, D \neq 0$, $\overline{N} = \{A;B;0\}$, $\overline{N} \perp Oz$ then we get an equation in a form $Ax + By + D = 0$ and we have a plane α that is parallel to the axis Oz.

3. If $A = D = 0$ and $\overline{N} = \{0;B;C\}$, then we get a plane equation in a form $By + Cz = 0$, it is a plane α that contain an axes Ox, $Ox \in \alpha$. Similarly, if $B = D = 0$, $\overline{N} = \{A;0;C\}$ then $Oy \in \alpha$; if $C = D = 0$, $\overline{N} = \{A;B;0\}$ then $Oz \in \alpha$.

4. If $D = 0$, $A = B = 0$, $\overline{N} = \{0;0;C\}$, $\overrightarrow{N} \perp Oz$, then a plane has this form of an equation $Cz + D = 0$, it is an equation of a plane that is parallel to the plane xOy and perpendicular to the axis Oz. If

$D=0$, $A=C=0$, $\overrightarrow{N}=\{0;B;0\}$, $\overrightarrow{N}\perp Oy$ then $\alpha\,//\,Oxz$ and $\alpha\perp Oy$. If $D=0$, $B=C=0$, $\overrightarrow{N}=\{A;0;0\}$, $\overrightarrow{N}\perp Ox$ then $\alpha\,//\,yOz$ and $\alpha\perp Ox$.

Example 7.1 Compose the equation of a plane that is perpendicular Oz and passes through a point $P(1,-2,3)$.

Solution. The equation of a plane, that is perpendicular to the axis Oz, has a form: $Cz+D=0$. We know that this plane passes thought the given point, $P(1,-2,3)$, in this case, the point coordinates satisfy to the plane equation; substitute the coordinates of the point in the equation and obtain:

$$C\cdot 3+D=0 \Rightarrow D=-3C.$$

Thus, $Cz-3C=0$, we can divide both side by C ($C\neq 0$) and get answer

$$z-3=0 \text{ or } z=3.$$

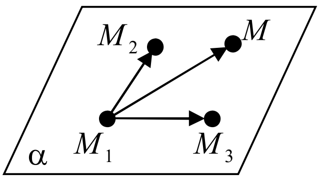


Figure 7.2

Write an equation for the plane containing points $M_1(x_1;y_1;z_1)$, $M_2(x_2;y_2;z_2)$, $M_3(x_3;y_3;z_3)$. We choose an arbitrary point $M(x;y;z)$ belonging to the desired plane (Figure 7.2) and compose vectors $\overrightarrow{M_1M_2}$, $\overrightarrow{M_1M}$, $\overrightarrow{M_1M_3}$ so they belong to the plane and their mixed product is zero. Let's take advantage of this fact and get new plane equation

$$(\overrightarrow{M_1M_2},\overrightarrow{M_1M},\overrightarrow{M_1M_3})=0,\quad \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0.$$

Example 7.2 Compose the equation of a plane that passes thought points $M_1(1;3;-2)$, $M_2(4;-5;6)$, $M_3(-3;1;2)$

Solution.

$$\begin{vmatrix} x-1 & y-3 & z-(-2) \\ 4-1 & -5-3 & 6-(-2) \\ -3-1 & 1-3 & 2-(-2) \end{vmatrix} = 0; \quad \begin{vmatrix} x-1 & y-3 & z+2 \\ 3 & -8 & 8 \\ -4 & -2 & 4 \end{vmatrix} = 0;$$

$$(x-1)\begin{vmatrix} -8 & 8 \\ -2 & 4 \end{vmatrix} - (y-3)\begin{vmatrix} 3 & 8 \\ -4 & 4 \end{vmatrix} + (z+2)\begin{vmatrix} 3 & -8 \\ -4 & -2 \end{vmatrix} = 0;$$

$$(x-1)(-32-(-16)) - (y-3)(12-(-32)) + (z+2)(-6-32) = 0;$$

$$-16(x-1) - 44(y-3) - 38(z+2) = 0 \;\Big|\; :(-2);$$

$$8(x-1) + 22(y-3) + 19(z+2) = 0;$$

$$8x - 8 + 22y - 66 + 19z + 38 = 0;$$

$$8x + 22y + 19z - 36 = 0.$$

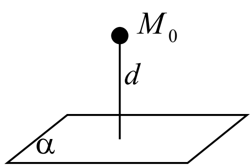


Figure 7.2

Now, that we can write an equation for a plane, we can use the equation to find the distance d between a point M_0 and the plane α (Figure7.2). It is defined as the shortest possible distance from M_0 to a point on the plane.

Just as we find the two-dimensional distance between a point and a line by calculating the length of a line segment perpendicular to the line, we find the three-dimensional distance between a point and a plane by calculating the length of a line segment perpendicular to the plane. The distance between a point

$M_0(x_0;y_0;z_0)$ and the plane $\alpha : Ax + By + Cz + D = 0$ can be calculated by this formula:

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

In addition to finding the equation of the line of intersection between two planes, we may need to find the angle formed by the intersection of two planes. For example, builders constructing a house need to know the angle where different sections of the roof meet to know whether the roof will look good and drain properly. We can use normal vectors to calculate the angle between the two planes. We can do this because the angle between the normal vectors is the same as the angle between the planes.

The angle between the planes $A_1x + B_1y + C_1z + D_1 = 0$ $A_2x + B_2y + C_2z + D_2 = 0$ is determined by the formula:

$$\cos\varphi = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

The angle between two planes can be

1) when $\varphi = 0$ then two planes are parallel and their normal vectors are parallel (Figure 7.3). So, the necessary and sufficient conditions for the parallelism of two planes is

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2};$$

2) the necessary and sufficient conditions for the coincidence of two planes is

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2};$$

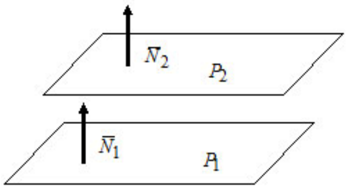


Figure 7.3

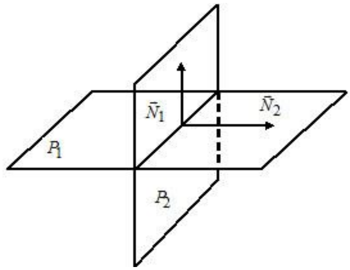


Figure 7.4

3) when $\varphi = 90$ then two planes are perpendicular and their normal vectors are perpendicular too (Figure 7.4). So, the necessary and sufficient conditions for the perpendicularity of two planes is

$$A_1A_2 + B_1B_2 + C_1C_2 = 0 .$$

Example 7.3 Compose the equation of a plane passing through points $P(1;-1;2)$ and $Q(3;1;2)$, and it is perpendicular to the plane $4x - 5y + 3z - 2 = 0$ (Figure 7.5)

Solution (the first way): let us consider the normal vector $\vec{N}$ of the desired plane to be a vector product of vectors $\vec{PQ}$ i $\vec{N}_1$. Find the coordinates of the vector $\vec{PQ}$

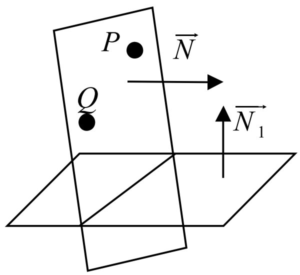


Figure 7.5

$$\vec{PQ} = (3-1; 1-(-1); 2-2) = (2;2;0),$$

$$\vec{PQ} \times \vec{N}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 0 \\ 4 & -5 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & 0 \\ -5 & 3 \end{vmatrix} -$$

$$-\vec{j}\begin{vmatrix} 2 & 0 \\ 4 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 2 \\ 4 & -5 \end{vmatrix} = 6\vec{i} - 6\vec{j} - 18\vec{k},$$

thus $\vec{N} = (6;-6;-18)$.

We use the equation of the plane passing through a given point and perpendicular to the vector (scalar equation)

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

and we have: $A=6$, $B=-6$, $C=-18$, $x_0=1$, $y_0=-1$, $z_0=2$

$$6(x-1)-6(y+1)-18(z-2)=0\,;$$

$$6x-6y-18z+24=0,\ \text{then}\ x-y-3z+4=0\,.$$

Solution (the second way): use the equation of the plane passing through a given point P and perpendicular to the vector:

$$A(x-1)+B(y+1)+C(z-2)=0\,.$$

Also this plane passing through a given point Q, therefore, its coordinates satisfy the plane equation:

$$A(3-1)+B(1+1)+C(2-2)=0\,,$$

whence we have $2A+2B=0$ or $A+B=0$.

We use the condition of perpendicularity of two planes:

$$4A-5B+3C=0\,.$$

Find A and B from the equations system:

$$\begin{cases} A+B=0 \\ 4A-5B+3C=0 \end{cases} \Rightarrow \begin{cases} A+B=0 \\ 4A-5B=-3C \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 \\ 4 & -5 \end{vmatrix}=-5-4=-9\,,$$

$$\Delta_A=\begin{vmatrix} 0 & 1 \\ -3C & -5 \end{vmatrix}=0-(-3C)=3C\,,$$

$$\Delta_B=\begin{vmatrix} 1 & 0 \\ 4 & -3C \end{vmatrix}=-3C-0=-3C\,,$$

$$A = \frac{\Delta_A}{\Delta} = \frac{3C}{-9} = -\frac{C}{3},\ B = \frac{\Delta_B}{\Delta} = \frac{-3C}{-9} = \frac{C}{3}.$$

Let's denote $C = 3t$, then $A = -t$, $B = t$. Substitute A, B, C at the a plane equation

$$A(x-1) + B(y+1) + C(z-2) = 0,$$

$$-t(x-1) + t(y+1) + 3t(z-2) = 0 \ \Big|: (-t),$$

$$(x-1) - (y+1) - 3(z-2) = 0 \ \Big|: (-t),$$

$$x - 1 - y - 1 - 3z + 6 = 0,$$

$$x - y - 3z + 4 = 0$$

Example 7.4 Find the intersection point of three planes:

$2x - 4y + 3z - 1 = 0$, $3x - y + 5z - 2 = 0$, $4x + 3y + 4z = 0$.

Solution. Make a system of plane equations and solve it using the Cramer's rule:

$$\Delta = \begin{vmatrix} 2 & -4 & 3 \\ 3 & -1 & 5 \\ 4 & 3 & 4 \end{vmatrix} = -8 + 27 - 80 + 12 + 48 - 30 = -31;$$

$$\Delta_x = \begin{vmatrix} 1 & -4 & 3 \\ 2 & -1 & 5 \\ 0 & 3 & 4 \end{vmatrix} = -4 + 18 + 0 + 0 + 32 - 15 = 31;$$

$$\Delta_y = \begin{vmatrix} 2 & 1 & 3 \\ 3 & 2 & 5 \\ 4 & 0 & 4 \end{vmatrix} = 16 + 0 + 20 - 24 - 12 - 0 = 0;$$

$$\Delta_z = \begin{vmatrix} 2 & -4 & 1 \\ 3 & -1 & 2 \\ 4 & 3 & 0 \end{vmatrix} = 0+9-32+4+0-12 = -31.$$

$$x = \frac{\Delta_x}{\Delta} = \frac{31}{-31} = -1,\ y = \frac{\Delta_y}{\Delta} = \frac{0}{-31} = 0,\ x = \frac{\Delta_x}{\Delta} = \frac{-31}{-31} = 1.$$

The intersection point of three planes is $M(-1;0;1)$.

The relationship between two planes in space has also two possibilities: the two distinct planes are parallel or they intersect. When two planes intersect, the intersection is a line (Figure 7.6).

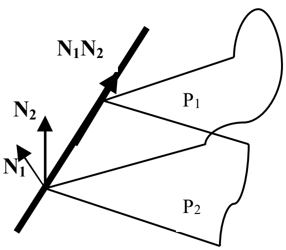


Figure 7.6

So, the general equation of a line in a space can be written in a form

$$\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases},$$

$\overrightarrow{N}_1 = \{A_1, B_1, C_1\}$ is a normal vector of the first plane P_1; $\overrightarrow{N}_2 = \{A_2, B_2, C_2\}$ is a normal vector of the second plane P_2.

By now, we are familiar with writing equations that describe a line in two dimensions. To write an equation for a line, we must know two points on the line, or we must know the direction of the line and at least one point through which the line passes. In two dimensions, we use the concept of slope to describe the orientation, or direction, of a line. In three dimensions, we describe the direction of a line using a vector parallel to the line. In this section, we examine how to use equations to describe lines and planes in space.

As in two dimensions, we can describe a line in space using a point on the line and the direction of the line, or a parallel vector, which we call the direction vector. It is a vector $\vec{s}$ that has coordinates $k, l, m,$ they are call *the direction coordinates***.**

The canonic equation of a line in a space looks like this

$$\frac{x-x_0}{k}=\frac{y-y_0}{l}=\frac{z-z_0}{m},$$

where $\vec{s}=(k,l,m)$ is a direction vector, $M_0(x_0,y_0,z_0)$ is a point on this line.

After some transformations we can rewrite in a parametric equations

$$\begin{cases} x = x_0 + kt \\ y = y_0 + lt \\ z = z_0 + mt \end{cases}.$$

Example 7.4 Make the parametric equations of a line, if we have her canonic equation $\frac{x+1}{5}=\frac{y+4}{-2}=\frac{z-3}{-3}$.

Solution. Equate each of the relations $\frac{x+1}{5}$, $\frac{y+4}{-2}$, $\frac{z-3}{-3}$ to the parameter t, which remains unchanged

$$\frac{x+1}{5}=t,\ \frac{y+4}{-2}=t,\ \frac{z-3}{-3}=t,$$

and express variables *x, y, z.*

We get

$$x=5t-1,\ y=-2t-4,\ z=-3t+3.$$

It is the parametric equations of a line.

However, we use an equation of a line passes thought two point in a space, it is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1},$$

where (x_1,y_1,z_1) and (x_2,y_2,z_2) are coordinates of points M_1 and M_2 respectively, that belong to the line.

The angel between two lines can be found by the formula

$$\cos\varphi = \frac{k_1k_2 + l_1l_2 + m_1m_2}{\sqrt{k_1^{\ 2} + l_1^{\ 2} + m_1^{\ 2}}\sqrt{k_2^{\ 2} + l_2^{\ 2} + m_2^{\ 2}}},$$

where (k_1, l_1, m_1) is a normal vector of the first line, (k_2, l_2, m_2) is a normal vector of the second line.

The parallel condition of two lines is

$$\overrightarrow{s_1} \parallel \overrightarrow{s_2} \Rightarrow \frac{k_1}{k_2} = \frac{l_1}{l_2} = \frac{m_1}{m_2} = \lambda .$$

The perpendicularity condition of two lines is

$$\overrightarrow{s_1} \perp \overrightarrow{s_2} \Rightarrow k_1k_2 + l_1l_2 + m_1m_2 = 0 .$$

The angel between a line and a plane in a space can be found by the formula

$$\sin\varphi = \left| \frac{Al + Bm + Cn}{\sqrt{A^2 + B^2 + C^2} \cdot \sqrt{l^2 + m^2 + n^2}} \right| ,$$

it is a sine of an angel φ between a line a and a plane α in a space, where A, B, C are coordinates of a normal vector $\overrightarrow{N}$ of a plane α; *k, l, m* are coordinates of a direction vector $\vec{s}$ of a line a.

If a line and a plane in a space are parallel then we have a *parallel condition*:

$$Ak + Bl + Cm = 0,$$

and if a line and a plane in a space are perpendicular then we have a *perpendicularity condition*:

$$\frac{A}{k} = \frac{B}{l} = \frac{C}{m}.$$

Example 7.6 Check that the line $\frac{x-2}{2} = \frac{y-3}{1} = \frac{z+1}{3}$ belongs to the plane $x + y - z - 6 = 0$.

Solution. We apply the condition of belonging of a line to a plane:

$$\begin{cases} 1\cdot 2+1\cdot 3-1\cdot(-1)-6=0 \\ 1\cdot 2+1\cdot 1-1\cdot 3=0 \end{cases}.$$

It is performed, therefore, this line belongs to the plane.

Example 7.7 Find the plane equation, which passes thought point $P(1;2;-1)$ and perpendicular to a line

$$\frac{x-3}{1}=\frac{y-2}{-3}=\frac{z+1}{4}.$$

Solution. Write the equation of the desired plane, applying the equation of the plane passing through this point:

$$A(x-1)+B(y-2)+C(z+1)=0.$$

Applying the condition of perpendicularity of the line and the plane, we replace the values A, B, C by proportional values *k, l, m* from the line equation

$$\frac{x-3}{1}=\frac{y-2}{-3}=\frac{z+1}{4},$$

and we get:

$$1(x-1)-3(y-2)+4(z+1)=0.$$

After simplification we have the equation of the desired plane:

$$x-3y+4z+9=0.$$

Lecture 8
Second order curves: circle, ellipse, hyperbola, parabola

The ***second order line*** is described by a second order equation, the general form of which is

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

where A, B, C, D, E, F are constant coefficients, moreover, at least one of the numbers A, B, C is non-zero, i.e. $A^2 + B^2 + C^2 \neq 0$.

There are four types of second-order lines: a circle, an ellipse, a hyperbola, and a parabola.

A ***circle*** is a set of all points in a plane that are at a given distance from a given point, the ***centre***. The distance between any of the points and the centre is called the ***radius*** (Figure 8.1).

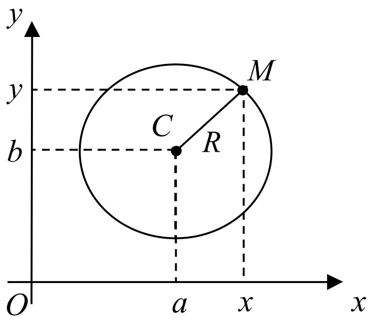


Figure 8.1

The circle is described by the equation

$$(x-a)^2 + (y-b)^2 = R^2,$$

where $C(a,b)$ is the center and R is the radius of the circle.

Equation of a circle centered at the origin

$$x^2 + y^2 = R^2$$

is known as the ***standard form*** of the equation of a circle.

Example 8.1 The circle is given by the equation

$$x^2 + y^2 - 6x - 4y + 4 = 0.$$

Find the radius and the coordinates of the center.

Solution. Transform the quadratic polynomial on the left-hand side of the equation by adding and subtracting the corresponding constants to complete the perfect squares:

$$\left(x^2-6x\right)+\left(y^2-4y\right)+4=0,$$

$$\left(\left(x^2-6x+9\right)-9\right)+\left(\left(y^2-4y+4\right)-4\right)+4=0,$$

$$\left(x-3\right)^2-9+\left(y-2\right)^2-4+4=0,$$

$$\left(x-3\right)^2+\left(y-2\right)^2=9.$$

Then the given equation is reduced to the form which describes the circle centered at the point $C(3,2)$ with radius $R=3$.

An ***ellipse*** (Figure 8.2) is a curve in a plane such that the sum of the distances to the two fixed points F_1 and F_2 is constant and equal to $2a$ for every point on the curve.

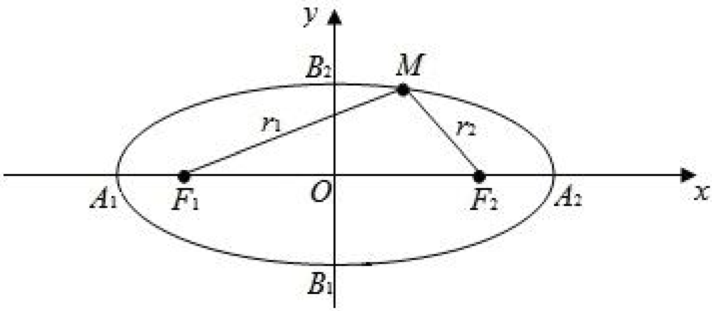


Figure 8.2

The fixed points $F_1(-c;0)$ and $F_2(c;0)$ are the ***foci*** of the ellipse. The line through the foci is the ***focal axis***. The point on the focal axis midway between the foci is the ***center***. The points $A_1(-a;0)$, $A_2(a;0)$, $B_1(0;-b)$, $B_2(0;b)$ where the ellipse intersects coordinate axes are the ***vertices*** of the ellipse. A line segment with endpoints on an ellipse is a ***chord*** of the ellipse. The chord lying on the focal axis is the ***major axis*** of the ellipse. The chord through the center perpendicular to the focal axis is the ***minor axis*** of the ellipse. The length of the major axis is $A_1A_2 = 2a$, and of the minor axis is $B_1B_2 = 2b$. The number a is the ***semimajor axis***, and b is the ***semiminor axis***.

The shape of an ellipse (how "elongated" it is) is represented by its ***eccentricity***, which for an ellipse can be any number from 0 (the limiting case of a circle) to 1:

$$\varepsilon = \frac{c}{a}.$$

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (b^2 = a^2 - c^2 > 0)$$

is the ***standard form*** of the equation of an ellipse centered at the origin with the x-axis as its focal axis.

An ellipse centered at the origin with the y-axis as its focal axis has an equation of the form

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

Example 8.2 Find the vertices, the foci and the eccentricity of the ellipse $4x^2 + 16y^2 - 64 = 0$.

Solution. Dividing both sides of the equation by 64 yields the standard form:

$$4x^2 + 16y^2 = 64 \Big| : 64, \quad \frac{4x^2}{64} + \frac{16y^2}{64} = 1,$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$

Because the larger number is the denominator of x^2, the focal axis is on the x-axis. So, $a^2 = 16$, $b^2 = 4$ and

$c^2 = a^2 - b^2 = 16 - 4 = 12$. Therefore, $a = 4$, $b = 2$ and $c = \sqrt{12} = 2\sqrt{3}$.

Thus, the vertices are $A_1(-4;0)$, $A_2(4;0)$, $B_1(0;-2)$, $B_2(0;2)$, the foci are $F_1(-2\sqrt{3};0)$, $F_2(2\sqrt{3};0)$ and the eccentricity is $\varepsilon = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$.

Example 8.3 Find an equation of the ellipse with the eccentricity $\varepsilon = 0,5$ whose semimajor axis is on the x-axis and has length 8.

Solution. The semimajor axis is $a = 8$. The eccentricity of the ellipse is $\varepsilon = \frac{c}{a} = 0,5$, so $c = a \cdot 0,5 = 8 \cdot 0,5 = 4$. Using $b^2 = a^2 - c^2$, we have $b^2 = 8^2 - 4^2 = 64 - 16 = 48$. So the standard form of the equation for ellipse is

$$\frac{x^2}{64} + \frac{y^2}{48} = 1.$$

A ***hyperbola*** (Figure 8.3) is the set of all points in a plane whose distances from two fixed points F_1 and F_2 in the plane have a constant difference equals $2a$.

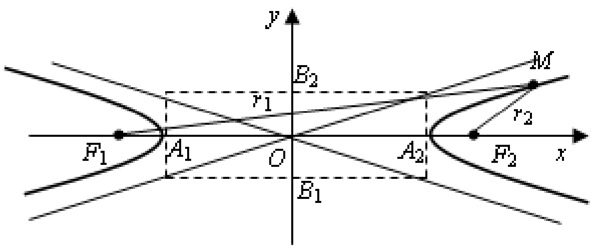


Figure 8.3

The fixed points $F_1(-c;0)$ and $F_2(c;0)$ are the ***foci*** of the hyperbola. The line through the foci is the ***focal axis***. The point on the focal axis midway between the foci is the ***center***. The points $A_1(-a;0)$, $A_2(a;0)$ where the hyperbola intersects its focal axis are the ***vertices*** of the hyperbola. A line segment with endpoints on a hyperbola is a ***chord*** of the hyperbola. The chord lying on the focal axis connecting the vertices is the ***transverse axis*** of the hyperbola. The length of the transverse axis is $2a$. The line segment of length $2b$ that is perpendicular to the focal axis and that has the center of the hyperbola as its midpoint is the ***conjugate axis*** of the hyperbola. The number a is the ***semitransverse axis***, and b is the ***semiconjugate axis***. Notice that the hyperbola has two ***branches***.

The shape of a hyperbola is represented by its ***eccentricity***. For a hyperbola the eccentricity $\varepsilon > 1$:

$$\varepsilon = \frac{c}{a}.$$

The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ \ (b^2 = c^2 - a^2 > 0)$$

is the ***standard form*** of the equation of a hyperbola centered at the origin with the x-axis as its focal axis. This hyperbola has two ***asymptotes*** $y = \pm\frac{b}{a}x$.

A hyperbola centered at the origin with the y-axis as its focal axis has an equation of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1,$$

and two ***asymptotes*** $y = \pm\frac{a}{b}x$.

Example 8.4 Find an equation of a hyperbola, if the distance between its foci is equal to 26, and the eccentricity is $\varepsilon = 13/12$.

Solution. Since the distance between the foci of a hyperbola is $2c = 26$, then $c = 13$. The eccentricity of the hyperbola is $\varepsilon = \dfrac{c}{a}$, so $a = \dfrac{c}{\varepsilon} = \dfrac{13}{13/12} = 13 \cdot \dfrac{12}{13} = 12$. Using $b^2 = c^2 - a^2$, we have $b^2 = 13^2 - 12^2 = 25$. So the standard form of the equation for hyperbola is

$$\frac{x^2}{144} - \frac{y^2}{25} = 1.$$

Example 8.5 Find an equation of a hyperbola with asymptotes $y = \pm 2x$, if the distance between its foci is equal to 10.

Solution. Since the distance between the foci of a hyperbola is $2c = 10$, then $c = 5$. From the equations of asymptotes $y = \pm \dfrac{b}{a} x = \pm 2x$ we have $\dfrac{b}{a} = 2$, so $b = 2a$. Using $b^2 = c^2 - a^2$, we have

$(2a)^2 = 5^2 - a^2$, $4a^2 = 5^2 - a^2$, $4a^2 + a^2 = 5^2$, $5a^2 = 25$, $a^2 = 5$,

$$a = \sqrt{5},\ b = 2a = 2\sqrt{5}.$$

So the standard form of the equation for hyperbola is

$$\frac{x^2}{\left(\sqrt{5}\right)^2} - \frac{y^2}{\left(2\sqrt{5}\right)^2} = 1,\ \frac{x^2}{5} - \frac{y^2}{20} = 1.$$

A ***parabola*** is the set of all points in a plane equidistant from a particular line l_d (the ***directrix***) and a particular point F (the ***focus***) in the plane (Figure 8.4).

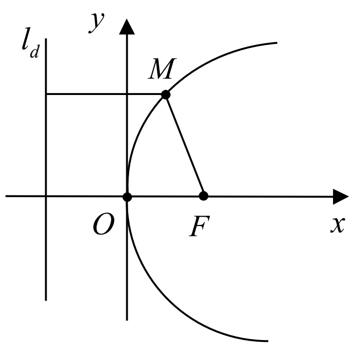


Figure 8.4

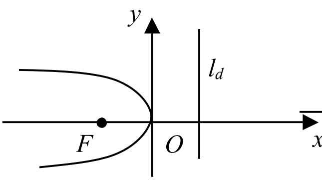
The line passing through the focus and perpendicular to the directrix is the ***(focal) axis*** of the parabola. The axis is the line of symmetry for the parabola. The point where the parabola intersects its axis is the ***vertex*** of the parabola. The vertex is located midway between the focus and the directrix and is the point of the parabola that is closest to both the focus and the directrix.

The general equation is complicated because of the choice of a general point and a general line. By an appropriate choice of axes this equation can be simplified; but it will then represent only parabolas in special positions. For example, if axes are chosen so that the focus has coordinates $F\left(\frac{p}{2};0\right)$ and the equation of the directrix is $l_d : x = -\frac{p}{2}$ (Figure 8.4), then the ***standard form*** of the equation of such parabola is

$$y^2 = 2px.$$

The ***eccentricity*** of the parabola is equal to one $\varepsilon = 1$.

Some more cases are shown at the figures below.

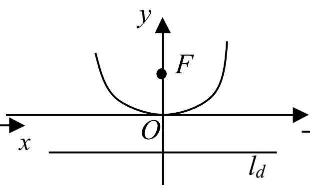


$y^2 = -2px$,

$l_d : x = \frac{p}{2}$,

$F\left(-\frac{p}{2};0\right)$

Figure 8.5

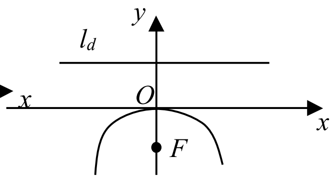


$x^2 = 2py$,

$l_d : y = -\frac{p}{2}$,

$F\left(0;\frac{p}{2}\right)$

Figure 8.6



$x^2 = -2py$,

$l_d : y = \frac{p}{2}$,

$F\left(0;-\frac{p}{2}\right)$

Figure 8.7

Example 8.6 Find the focus and the equation of the directrix of the parabola $y^2 + 4x = 0$.

Solution. The standard form of the given equation is $y^2 = -4x$.
The coefficient of x is $-2p = -4$, $p = 2$.

So, the focus is $F\left(-\frac{p}{2};0\right) = F(-1;0)$.

The directrix is the line $x = \frac{p}{2}$, i.e. $x = 1$.

In the Cartesian coordinate system to define the point with the coordinates (x, y) we start from the origin and then move x units horizontally followed by y units vertically.

However, this is not the only one way to define a point in two dimensional space. The ***polar coordinate system*** is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a ***reference point*** and an angle from a ***reference direction***.

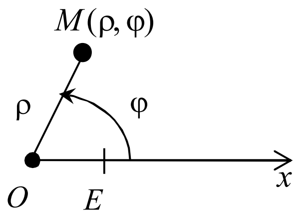


Figure 8.8

The reference point O (analogous to the origin of a Cartesian coordinate system) is called the ***pole***, and the ray from the pole in the reference direction is the ***polar axis***. The distance $\rho = OM$ from the pole is called the ***radial coordinate*** or ***radius***, and the angle φ is called the ***angular coordinate***, ***polar angle***, or ***azimuth*** (Figure 8.8).

Polar coordinates (ρ, φ) are connected with rectangular coordinates (x, y) by the relations:

$$x = \rho\cos\varphi,\ y = \rho\sin\varphi, \text{ (remember that } x^2 + y^2 = \rho^2\text{)}.$$

The Cartesian coordinates (x, y) can be converted to polar coordinates (ρ, φ) by the following relations:

$$\rho = \sqrt{x^2 + y^2}\ ,\ tg\varphi = \frac{y}{x}\ ,\ \sin\varphi = \frac{y}{\sqrt{x^2 + y^2}}\ ,\ \cos\varphi = \frac{x}{\sqrt{x^2 + y^2}}\ .$$

The equation defining an algebraic curve expressed in polar coordinates is known as a ***polar equation***. In many cases, such an equation can be specified more simply by defining ρ as a function of φ. Then the resulting curve consists of points in a form $(\rho(\varphi), \varphi)$ and it can be regarded as the graph of the polar function ρ respecting to φ.

Example 8.7 The curve graph given by the equation

$$\rho = 1 - \cos 2\varphi\ .$$

Solution. To draw a line in a polar coordinate system, we will compile a table of values of the polar radius ρ for certain values of the polar angle φ:

$$\varphi = 0\ ,\ \rho = 1 - \cos(2 \cdot 0) = 1 - \cos 0 = 1 - 1 = 0\ ,$$

$$\varphi = \frac{\pi}{8}\ ,\ \rho = 1 - \cos\left(2 \cdot \frac{\pi}{8}\right) = 1 - \cos\frac{\pi}{4} = 1 - \frac{\sqrt{2}}{2} \approx 0,29\ ,\ \ldots$$

φ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
ρ	0	0,29	1	1,7	2	1	0	1	2	1	0

Mark the points and construct a line (Figure 8.9).

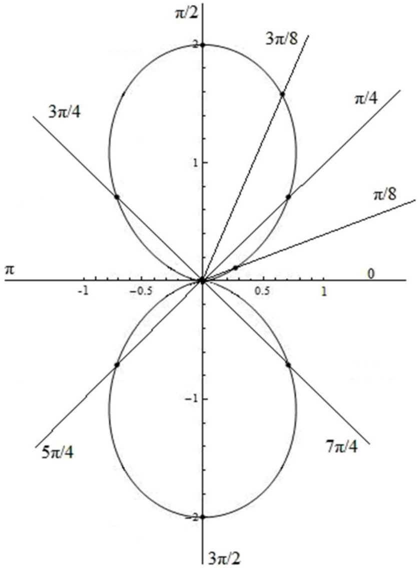


Figure 8.9

Remember, that sometimes in a some complicated tasks more easy to use polar system coordinate, that help to simplify the process of a solution, namely to replaced the Cartesian coordinate to the polar.

Also, in addition to the polar coordinate system, a cylindrical or spherical coordinate system is sometimes used, which also significantly simplify the process of finding a solution.

We will talk about these coordinate systems later.

You can find out more about these coordinate systems by yourself using the list of literature suggested at the end of the synopsis.

Lecture 9
Surfaces of the second order

The surface of the second order is called the set of points of space, the coordinates of which satisfy the algebraic equation of the second degree

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fxz + 2Gx + 2Hy + 2Kz + L = 0. \qquad (9.1)$$

where at least one of the coefficients A, B, C, D, E, F is non-zero. This equation can define a sphere, an ellipsoid, a hyperboloid (single-cavity or double-cavity), a paraboloid (elliptical or hyperbolic), a cone, a cylinder (elliptical, hyperbolic or parabolic), as well as a degenerate surface, an empty plane, a couple of planes). Due to the parallel transfer and rotation of the coordinate system, equation (9.1) can be reduced to the canonical form. The shape and location of surfaces are studied by the method of cross sections (Appendix A). To do this, cross the surface with planes parallel to the coordinate planes, and determine the type of curve obtained at this intersection

Canonical equation of an ellipsoid with semiaxes a, b, c is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (9.2)$$

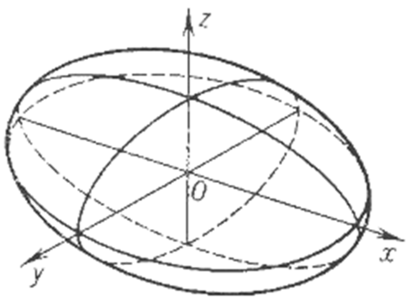


Figure 9.1

The surface (Figure 9.1) is symmetrical about the coordinate axes and coordinate planes. The center of symmetry is at the origin. We use the cross-sectional method to determine the shape of the surface. Cut the surface with a:

plane $x = h$

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{h^2}{a^2};$$

1) if $|h|<a$ then $1-\frac{h^2}{a^2}=H^2>0$ and in this case in cross section will be an ellipse with axes Hb, Hc;

2) if $h=\pm a$ then we get two points $(a,0,0)$, $(-a,0,0)$;

3) if $|h|>a$ then we get $1-\frac{h^2}{a^2}<0$ and in this case the plane and the surface do not intersect.

If $a=b=c=R$ then the equation (9.1) transform in a sphere equation that has the centre at the origin point and radius R:

$$x^2+y^2+z^2=R^2 \tag{9.3}$$

Sphere equation that has the centre at the point $M_o(x_0, y_0, z_0)$ and radius $R>0$ is:

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2 \tag{9.4}$$

Hyperboloid is a quadric surface having a center not at infinity, and some of its plane sections hyperbolas. There are two kinds of hyperboloid, those of one and of two sheets. The hyperboloid of one sheet has a real intersection with every plane in space; that of two sheets has only imaginary intersections with some planes. In either case all the plane sections perpendicular to one of the axes are ellipses, and those perpendicular to either of the others are hyperbolas.

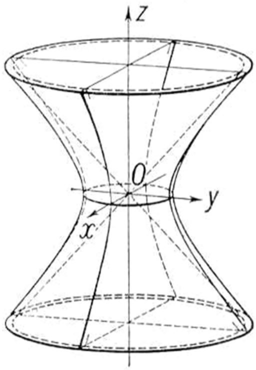


Figure 9.2

Canonical equation of a single-cavity hyperboloid (Figure 9.2) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ a,b,c>0 \qquad (9.5)$$

The surface is symmetrical with respect to the coordinate planes $x0z$ i $y0z$.

The origin is the center of symmetry. The cross sections of the surface equation (9.5) by planes $x=0$, $y=0$ are hyperbolas and we get:

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \qquad \frac{x^2}{a^2}-\frac{z^2}{c^2}=1.$$

The cross section of the surface by plane $z=h$ is an ellipse

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1+\frac{h^2}{a^2}=H^2 \Rightarrow \frac{x^2}{(aH)^2}+\frac{y^2}{(bH)^2}=1\cdot$$

A single-cavity hyperboloid belongs to linear surfaces. It can be constructed using two systems of straight lines.

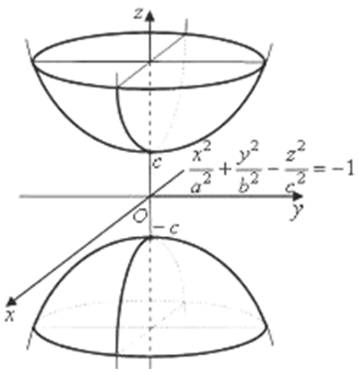


Figure 9.3

Canonical equation of a two-cavity hyperboloid is:

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ (a,b,c>0) \qquad (9.6)$$

The surface (Figure. 9.3) is symmetrical about the coordinate axes and coordinate planes. The center of symmetry is at the origin.

The cross sections of the surface (9.6) by the planes $x=0$ and $y=0$ are hyperbolas that have the following equations:

$$-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ (b,c>0);$$

$$-\frac{x^2}{a^2}+\frac{z^2}{c^2}=1,\ (a,c>0).$$

Consider the cross sections of the surface (9.6) by the plane $z=h$:

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{h^2}{a^2}-1;$$

1) if $|h|<c$ – then the section is an empty set;

2) if $|h|=c$ that at the section we get two points: $(0,0,c)$, $(0,0,-c)$;

3) if $|h|>c$ that at the section we get an ellipse:

$$\frac{x^2}{(aH)^2}+\frac{y^2}{(bH)^2}=1,\quad \left(H^2=\frac{h^2}{c^2}-1\right).$$

A conical surface is a surface described by a straight line (generating) passing through a fixed point S (the vertex of a cone) and a variable point M moving along a curve (a conical surface guide). If the guide of a conical surface is a curve of the second order, then the surface is called a cone of the second order.

The canonical equation of an elliptical cone of the second order is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,\ (a,b,c>0). \qquad (9.6)$$

The surface is symmetrical about the coordinate axes and coordinate planes. The center of symmetry is at the origin (Figure 9.4).

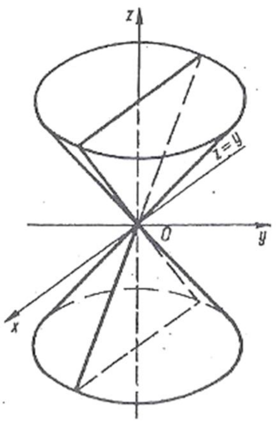


Figure 9.4

Cone cross sections by planes $x=0$ and $y=0$ are intersecting lines:

$$y = \pm bz/c\,,\; x = \pm az/c\,.$$

Surface sections by plane $z = h$ are ellipses:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{h^2}{c^2}\,,\quad \frac{x^2}{(aH)^2}+\frac{y^2}{(bH)^2}=1,\quad \left(H^2=\frac{h^2}{c^2}\right).$$

The equations of elliptical conical surfaces of the second order, the axes of which coincide with the axes $0x$, $0y$, have the form:

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,\ (a,b,c>0),\ \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,\ (a,b,c>0) \qquad (9.7)$$

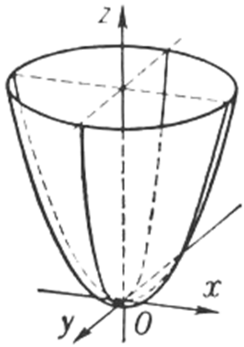


Figure 9.5

Canonical equation of an elliptical paraboloid is:

$$\frac{x^2}{p}+\frac{y^2}{q}=2z\,,\ (p,q>0) \qquad (9.8)$$

The surface is symmetrical about the coordinate planes $x0z$, $y0z$ and the axis $0z$ (Figure 9.5). Surface sections by planes $x = 0$, $y = 0$ are parabolas:

$$y^2 = 2qz\,,\quad x^2 = 2pz\,.$$

Surface sections by planes $z = h > 0$ are ellipses

$$\frac{x^2}{(\sqrt{2ph})^2}+\frac{y^2}{(\sqrt{2qh})^2}=1$$

Equation of hyperbolic paraboloid is:

$$\frac{x^2}{p}-\frac{y^2}{q}=2z\,,\ (p,q>0) \qquad (9.9)$$

The surface is symmetrical about the coordinate planes $x0z$, $y0z$ (Figure 9.6).

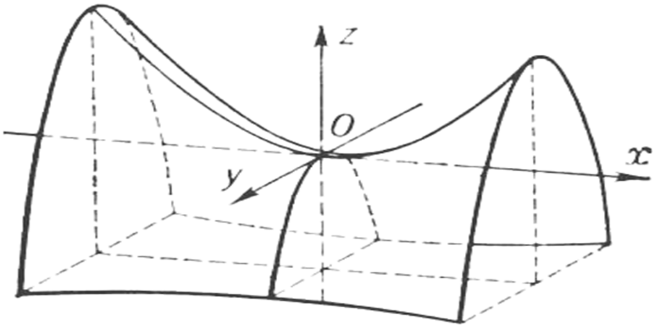


Figure 9.6

1. In the cross section of the surface by the plane $x = 0$ we obtain a parabola: $y^2 = -2qz$. The branches of the parabola are directed downwards.

2. In the cross section of the surface by the plane $y = 0$ we obtain a parabola: $x^2 = 2pz$. The branches of the parabola are directed upwards.

3. In the cross section of the surface by the plane $z = h$ we obtain a hyperbola: $\frac{x^2}{p} - \frac{y^2}{q} = 2h$. If $h > 0$ then the hyperbola branches intersect the axis $0x$; if $h < 0$ then the hyperbola branches intersect the axis $0y$. When $h = 0$ we get two straight lines intersecting at the origin point.

A cylindrical surface is a surface described by a straight line (generating) that moves parallel to itself along a given line (cylinder guide). If the guide of the cylinder lies in the plane $x0y$, and the generator is parallel to the axis $0z$, then the equation of the cylinder has the form: $F(x, y) = 0$ or $y = f(x)$. Similarly, $F(x, z) = 0$ or $z = f(x)$ is an equation of a cylindrical surface whose generator is parallel to the axis $0y$; $F(y, z) = 0$ or $z = f(y)$ is an equation of a cylindrical surface whose generator is parallel to the axis $0x$.

If the guide of a cylindrical surface is a second-order curve, then the surface is called a cylindrical surface of the second order.

According to the type of curve obtained in the cross section of the cylinder with a plane perpendicular to the generating, distinguish such second-order cylinders:

$x^2 + y^2 = R^2$ is an equation of a circular cylinder;

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an equation of a elliptical cylinder;

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is an equation of a hyperbolic cylinder;

$x^2 = 2py$ is an equation of a parabolic cylinder.

Similarly, we can write the equation of cylindrical surfaces of the second order, the product of which is parallel to the axes $0x$, $0y$.

At the figures below we can see circular (Figure 9.7) and hyperbolic (Figure 9.8) cylinders.

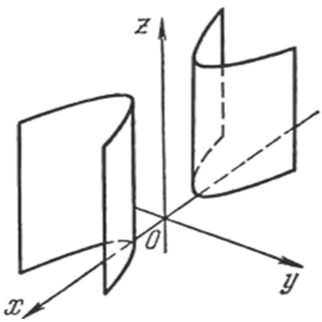


Figure 9.7

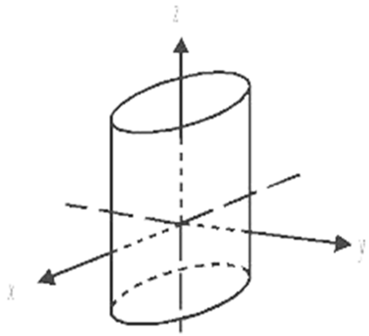


Figure 9.8

All graphs of surfaces of the second order with their equations are presented in Appendices B.

Lecture 10
Theory of Limits

The number A is called the ***limit of the function*** $y = f(x)$ when $x \to a$, if for all values of x that differ little enough from the number a, the corresponding values of the function $y = f(x)$ differ little enough from the number A:

$$\lim_{x \to a} f(x) = A .$$

If $x < a$ and $x \to a$, then we write conventionally $x \to a - 0$; similarly, if $x > a$ and $x \to a$, then we write $x \to a + 0$. The numbers

$$f(a-0) = \lim_{x \to a-0} f(x) \text{ and } f(a+0) = \lim_{x \to a+0} f(x)$$

are called, respectively, the ***limit*** of the function $f(x)$ ***from the left*** and the ***limit*** of the function $f(x)$ ***from the right*** at the point a (if these numbers exist).

For the existence of the limit of a function $f(x)$ as $x \to a$, it is necessary and sufficient to have the following equality:

$$f(a-0) = f(a+0) .$$

Example 10.1 Find the limits on the right and left of the function

$$f(x) = \arctan\frac{1}{x}$$

when $x \to 0$.

Solution:

$$f(+0) = \lim_{x \to +0}\left(\arctan\frac{1}{x}\right) = \arctan\frac{1}{+0} = \arctan(+\infty) = \frac{\pi}{2},$$

$$f(-0)=\lim_{x\to -0}\left(\arctan\frac{1}{x}\right)=\arctan\frac{1}{-0}=\arctan(-\infty)=-\frac{\pi}{2}.$$

Obviously, the function $f(x)$ in this case has no limit as $x\to 0$.

If the limits $\lim_{x\to a} f_1(x)$ and $\lim_{x\to a} f_2(x)$ exist, then the following theorems hold:

1) $\lim_{x\to a}\left[f_1(x)\pm f_2(x)\right]=\lim_{x\to a} f_1(x)\pm\lim_{x\to a} f_2(x)$,

2) $\lim_{x\to a}\left[f_1(x)\cdot f_2(x)\right]=\lim_{x\to a} f_1(x)\cdot\lim_{x\to a} f_2(x)$,

3) $\lim_{x\to a}\left[C\cdot f_1(x)\right]=C\cdot\lim_{x\to a} f_1(x)$,

4) $\lim_{x\to a}\frac{f_1(x)}{f_2(x)}=\frac{\lim_{x\to a} f_1(x)}{\lim_{x\to a} f_2(x)}$ ($\lim_{x\to a} f_2(x)\neq 0$).

Example 10.2 Compute $\lim_{x\to -2}\frac{x^2-x+4}{x^2+1}$.

Solution:

$$\lim_{x\to -2}\frac{x^2-x+4}{x^2+1}=\frac{\lim_{x\to -2}\left(x^2-x+4\right)}{\lim_{x\to -2}\left(x^2+1\right)}=$$

$$\frac{\lim_{x\to -2} x^2+\lim_{x\to -2} x+\lim_{x\to -2} 4}{\lim_{x\to -2} x^2+\lim_{x\to -2} 1}=\frac{\left(\lim_{x\to -2} x\right)^2+\lim_{x\to -2} x+\lim_{x\to -2} 4}{\left(\lim_{x\to -2} x\right)^2+\lim_{x\to -2} 1}=$$

$$=\frac{\left(\lim_{x\to -2} x\right)^2+\lim_{x\to -2} x+\lim_{x\to -2} 4}{\left(\lim_{x\to -2} x\right)^2+\lim_{x\to -2} 1}=\frac{\left(\lim_{x\to -2}(-2)\right)^2+\lim_{x\to -2}(-2)+\lim_{x\to -2} 4}{\left(\lim_{x\to -2}(-2)\right)^2+\lim_{x\to -2} 1}=$$

$$=\frac{(-2)^2+(-2)+4}{(-2)^2+1}=\frac{6}{5}.$$

The function $f(x)$ is called ***infinitesimal*** as $x \to a$ if

$$\lim_{x\to a} f(x) = 0 .$$

The function $f(x)$ is called ***infinitude*** as $x \to a$ if

$$\lim_{x\to a} f(x) = \infty .$$

Properties of infinitesimal and infinitude functions:

1) if $f(x)$ is infinitesimal function as $x \to a$, then $-f(x)$ is also infinitesimal one;

2) if $f_1(x)$ and $f_2(x)$ are infinitesimal functions as $x \to a$, then $f_1(x) \pm f_2(x)$ is also infinitesimal one;

3) if $f_1(x)$ and $f_2(x)$ are infinitude functions as $x \to a$, then $f_1(x) + f_2(x)$ and $f_1(x) \cdot f_2(x)$ are also infinitude ones;

4) if $\lim_{x\to a} f_1(x) = b = \text{const}$, $\lim_{x\to a} f_2(x) = \infty$, then

$$\lim_{x\to a}\left[f_1(x) + f_2(x)\right] = b + \infty = \infty , \quad \lim_{x\to a}\left[f_1(x) \cdot f_2(x)\right] = b \cdot \infty = \infty ,$$

$$\lim_{x\to a}\left(f_2(x)\right)^{f_1(x)} = \infty^b = \infty , \quad \lim_{x\to a} \sqrt[f_1(x)]{f_2(x)} = \sqrt[b]{\infty} = \infty ,$$

$$\lim_{x\to a} \frac{f_1(x)}{f_2(x)} = \frac{b}{\infty} = 0 ;$$

5) if $\lim_{x\to a} f_1(x) = b = \text{const}$, $\lim_{x\to a} f_2(x) = 0$, then

$$\lim_{x\to a} \frac{f_1(x)}{f_2(x)} = \frac{b}{0} = \infty .$$

We will now consider the cases where, for some assigned value of x, the numerator and denominator are both zero or both infinity. The fraction is then said to be ***indeterminate***.

During computing of the limit of two integral polynomials ratio as $x \to \infty$ and getting the indeterminate form $\left[\frac{\infty}{\infty}\right]$, it is necessary, firstly, to divide both terms of the ratio by x^n, where n is the highest degree of these polynomials. A similar procedure is also possible in many cases for fractions containing irrational terms.

Example 10.3 Compute $\lim\limits_{x\to\infty}\frac{x^2-3x+5}{2x^2-3}$.

Solution:

$$\lim_{x\to\infty}\frac{x^2-3x+5}{2x^2-3}=\frac{\infty^2-3\cdot\infty+5}{2\cdot\infty^2-3}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{5}{x^2}}{\frac{2x^2}{x^2}-\frac{3}{x^2}}=\lim_{x\to\infty}\frac{1-\frac{3}{x}+\frac{5}{x^2}}{2-\frac{3}{x^2}}=$$

$$=\frac{1-\frac{3}{\infty}+\frac{5}{\infty^2}}{2-\frac{3}{\infty^2}}=\frac{1-0+0}{2-0}=\frac{1}{2}.$$

Example 10. 4 Compute $\lim\limits_{x\to\infty}\frac{x^2-3x+5}{x-3}$.

Solution:

$$\lim_{x\to\infty}\frac{x^2-3x+5}{x-3}=\frac{\infty^2-3\cdot\infty+5}{\infty-3}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{5}{x^2}}{\frac{x}{x^2}-\frac{3}{x^2}}=\lim_{x\to\infty}\frac{1-\frac{3}{x}+\frac{5}{x^2}}{\frac{1}{x}-\frac{3}{x^2}}=$$

$$=\frac{1-\frac{3}{\infty}+\frac{5}{\infty^2}}{\frac{1}{\infty}-\frac{3}{\infty^2}}=\frac{1-0+0}{0-0}=\frac{1}{0}=\infty.$$

If $P(x)$ and $Q(x)$ are integral polynomials and $P(a)\neq 0$ or $Q(a)\neq 0$, then the limit of the rational fraction

$$\lim_{x\to a}\frac{P(x)}{Q(x)}$$

is obtained directly.

But if $P(a)=Q(a)=0$, then it is advisable to cancel the binomial $x=-a$ out of the fraction $\dfrac{P(x)}{Q(x)}$ once or several times. To do it, we can use the formulas of abridged multiplication:

1) $a^2-b^2=(a-b)(a+b)$,

2) $a^3\pm b^3=(a\pm b)\left(a^2\mp ab+b^2\right)$,

3) $ax^2+bx+c=a(x-x_1)(x-x_2)$, where x_1, x_2 are roots of the equation $ax^2+bx+c=0$ which can be found by using the discriminant:

$$D=b^2-4ac,\ x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}.$$

Example 10.5 Evaluate the following limits

$$\text{a) } \lim_{x\to -1}\frac{2x^2+x-1}{x^2-1};\ \text{b) } \lim_{x\to 3}\frac{2x^2-5x-3}{x^3-27}.$$

Solution:

$$\text{a) } \lim_{x\to -1}\frac{2x^2+x-1}{x^2-1}=\frac{2\cdot(-1)^2-1-1}{(-1)^2-1}=\frac{2-1-1}{1-1}=\left[\frac{0}{0}\right]=$$

Factorize the numerator and denominator and cancel:

$$x^2-1=(x-1)(x+1),\ 2x^2+x-1$$

$$D=1^2-4\cdot 2\cdot(-1)=1+8=9,$$

$$x_1=\frac{-1+\sqrt{9}}{2\cdot 2}=\frac{-1+3}{4}=\frac{2}{4}=\frac{1}{2},$$

$$x_2=\frac{-1-\sqrt{9}}{2\cdot 2}=\frac{-1-3}{4}=\frac{-4}{4}=-1,$$

$$2x^2+x-1=2\left(x-\frac{1}{2}\right)(x+1).$$

$$=\lim_{x\to -1}\frac{2\left(x-\frac{1}{2}\right)(x+1)}{(x-1)(x+1)}=\lim_{x\to -1}\frac{2\left(x-\frac{1}{2}\right)}{x-1}=\lim_{x\to -1}\frac{2x-1}{x-1}=\frac{2\cdot(-1)-1}{-1-1}=\frac{-2-1}{-1-1}=\frac{-3}{-2}=\frac{3}{2};$$

b) $\lim\limits_{x\to 3}\dfrac{2x^2-5x-3}{x^3-27}=\left|\dfrac{0}{0}\right|=$

$$=\left|\begin{array}{c} 2x^2-5x-3=2(x-3)(x+1/2) \\ D=1; x_1=3; x_2=-\dfrac{1}{2} \\ x^3-27=(x-3)\left(x^2+3x+9\right) \end{array}\right|=\lim_{x\to 3}\frac{2(x-3)(x+1/2)}{(x-3)\left(x^2+3x+9\right)}=$$

$$=\lim_{x\to 3}\frac{2(x+1/2)}{x^2+3x+9}=\frac{2\cdot 3+1}{9+9+9}=\frac{7}{27}.$$

To find the limit of an irrational expression, when one gets the indeterminate value $\left[\dfrac{0}{0}\right]$ or $[\infty-\infty]$, it is necessary to transfer the irrational term from the numerator to the denominator, or vice versa, from the denominator to the numerator.

Example 10.6 Compute $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+3x}-1}$.

Solution.

$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}=\frac{0}{\sqrt{1+3\cdot 0}-1}=\left[\frac{0}{0}\right]=$$

Multiply the numerator and denominator of the fraction under the limit sign by the conjugated expression of the denominator, i.e.

by $\sqrt{1+3x}+1$:

$$\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{\left(\sqrt{1+3x}-1\right)\left(\sqrt{1+3x}+1\right)}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{\left(\sqrt{1+3x}\right)^2-1^2}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{(1+3x-1)}=$$

$$=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}=\lim_{x\to 0}\frac{\sqrt{1+3x}+1}{3}=\frac{\sqrt{1+0}+1}{3}=\frac{2}{3}.$$

We have two fundamental limits that help us to simplify the limits calculation and they are frequently used.

Trigonometric functions:

$$\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=1,$$

and some useful consequences :

$$\lim_{\alpha\to 0}\frac{\tan\alpha}{\alpha}=1,\quad \lim_{\alpha\to 0}\frac{\arcsin\alpha}{\alpha}=1,\quad \lim_{\alpha\to 0}\frac{\arctan\alpha}{\alpha}=1,$$

$$\lim_{\alpha\to 0}\frac{\alpha}{\sin\alpha}=1,\quad \lim_{\alpha\to 0}\frac{\alpha}{\tan\alpha}=1,\quad \lim_{\alpha\to 0}\frac{\alpha}{\arcsin\alpha}=1,$$

$$\lim_{\alpha\to 0}\frac{\alpha}{\arctan\alpha}=1.$$

Exponential functions:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e\approx 2,72.$$

If you get the indeterminate value $\left[\frac{0}{0}\right]$ of the limit with trigonometric expressions it is necessary to factorize the numerator and denominator by using trigonometric formulas and cancel or apply the frequently used limits for trigonometric functions.

Note. Sometimes we need to use your school knowledge about the trigonometric functions which we can find at Appendices B-C.

Example 10.7 Compute $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}$.

Solution:

$$\lim_{x\to 0}\frac{\sin 3x}{x}=\frac{\sin 0}{0}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\sin 3x\cdot 3}{x\cdot 3}=3\cdot\lim_{x\to 0}\frac{\sin 3x}{3x}=3\cdot 1=3\,.$$

Example 9.8 Compute: a) $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{1-\cos 8x}$, b) $\lim\limits_{x\to\pi/4}\dfrac{ctg(\pi/4+x)}{4x-\pi}$.

Solution:

$$\text{a) } \lim_{x\to 0}\frac{1-\cos 4x}{1-\cos 8x}=\frac{1-\cos 0}{1-\cos 0}=\left[\frac{0}{0}\right]=,$$

firstly, we should use a trigonometric formula $\sin^2\alpha=\dfrac{1-\cos 2\alpha}{2}$ (look at Appendixe C), then frequently used limits for trigonometric functions:

$$=\left|1-\cos 2\alpha=2\sin^2\alpha\right|=\lim_{x\to 0}\frac{2\sin^2\frac{4x}{2}}{2\sin^2\frac{8x}{2}}=\lim_{x\to 0}\frac{\sin^2 2x}{\sin^2 4x}=\lim_{x\to 0}\frac{\sin 2x\cdot\sin 2x\cdot 2x\cdot 2x}{\sin^2 4x\cdot 2x\cdot 2x}=$$

$$=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{4x^2}{\sin^2 4x}=1\cdot 1\cdot\lim_{x\to 0}\frac{4x}{\sin 4x}\cdot\lim_{x\to 0}\frac{x}{\sin 4x}=$$

$$=1\cdot 1\cdot 1\cdot\lim_{x\to 0}\frac{x\cdot 4}{\sin 4x\cdot 4}=\lim_{x\to 0}\frac{x\cdot 4}{\sin 4x}\cdot\frac{1}{4}=1\cdot\frac{1}{4}=\frac{1}{4};$$

$$\text{b) } \lim_{x\to\pi/4}\frac{ctg(\pi/4+x)}{4x-\pi}=\left|\frac{0}{0}\right|=\left|\begin{matrix}u=x-\pi/4;\ x=\pi/4+u;\\ x\to\pi/4\ \Rightarrow\ u\to 0\end{matrix}\right|=$$

$$\lim_{u\to 0}\frac{ctg(\pi/4+u+\pi/4)}{4(\pi/4+u)-\pi}=\lim_{u\to 0}\frac{ctg(\pi/2+u)}{4u}=$$

$$= -\frac{1}{4} \cdot \lim_{u \to 0} \frac{tg\, u}{u} = -\frac{1}{4} \cdot 1 = -\frac{1}{4}.$$

When taking limits of the form

$$\lim_{x \to a} \left[f(x) \right]^{g(x)} = A,$$

one should bear in mind that:

1) if there are final limits

$$\lim_{x \to a} f(x) = B \text{ and } \lim_{x \to a} g(x) = C,$$

then $A = B^C$;

2) if lim $\lim_{x \to a} f(x) = B \neq 1$ and $\lim_{x \to a} g(x) = +\infty$, then

$$A = \begin{cases} 0, & B < 1; \\ \infty, & B > 1; \end{cases}$$

3) if $\lim_{x \to a} f(x) = B = 1$ and $\lim_{x \to a} g(x) = \infty$, then we get the indefinite value $\left[1^\infty \right]$ and should use frequently used limits for exponential functions.

Example 10.9 Compute $\lim_{x \to \infty} \left(\frac{2x+3}{x-1} \right)^{4x}$.

Solution:

$$f(x) = \frac{2x+3}{x-1}, \quad g(x) = 4x,$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{2x+3}{x-1} = \frac{2 \cdot \infty + 3}{\infty - 1} = \left[\frac{\infty}{\infty} \right] = \lim_{x \to \infty} \frac{\frac{2x}{x} + \frac{3}{x}}{\frac{x}{x} - \frac{1}{x}} = \lim_{x \to \infty} \frac{2 + \frac{3}{x}}{1 - \frac{1}{x}} = \frac{2 + \frac{3}{\infty}}{1 - \frac{1}{\infty}} = \frac{2+0}{1-0} = 2,$$

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} (4x) = 4 \cdot \infty = \infty.$$

Thus, we have the second case and $\lim\limits_{x\to\infty}\left(\dfrac{2x+3}{x-1}\right)^{4x}=2^{\infty}=\infty$, because $2>1$.

Example 10.10 Compute $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x-1}\right)^{4x}$.

Solution:

$$f(x)=\frac{x+3}{x-1},\quad g(x)=4x,$$

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{x+3}{x-1}=\frac{\infty+3}{\infty-1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\frac{x}{x}+\frac{3}{x}}{\frac{x}{x}-\frac{1}{x}}=\lim_{x\to\infty}\frac{1+\frac{3}{x}}{1-\frac{1}{x}}=\frac{1+\frac{3}{\infty}}{1-\frac{1}{\infty}}=\frac{1+0}{1-0}=1,$$

$$\lim_{x\to\infty}g(x)=\lim_{x\to\infty}(4x)=4\cdot\infty=\infty.$$

Thus, we have the third case: $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x-1}\right)^{4x}=\left[1^{\infty}\right]=$

and to find the limit we should use frequently used limits for exponential functions:

$$=\lim_{x\to\infty}\left(1+\left[\frac{x+3}{x-1}-1\right]\right)^{4x}=\lim_{x\to\infty}\left(1+\frac{x+3-(x-1)}{x-1}\right)^{4x}=\lim_{x\to\infty}\left(1+\frac{x+3-x+1}{x-1}\right)^{4x}=\lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{4x}=$$

$$=\lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{4x\cdot\frac{4}{x-1}\cdot\frac{x-1}{4}}=\lim_{x\to\infty}\left(\left(1+\frac{4}{x-1}\right)^{\frac{x-1}{4}}\right)^{4x\cdot\frac{4}{x-1}}=e^{\lim\limits_{x\to\infty}\frac{16x}{x-1}}=e^{\frac{16\cdot\infty}{\infty-1}}=e^{\left[\frac{\infty}{\infty}\right]}=$$

$$=e^{\lim\limits_{x\to\infty}\frac{\frac{16x}{x}}{\frac{x}{x}-\frac{1}{x}}}=e^{\lim\limits_{x\to\infty}\frac{16}{1-\frac{1}{x}}}=e^{\frac{16}{1-\frac{1}{\infty}}}=e^{\frac{16}{1-0}}=e^{16}.$$

Lecture 11
Derivative calculus

An *increment* of a variable quantity is any addition to its value, and is denoted by the symbol Δ written before this quantity. Thus Δx denotes an increment of x, Δy is an increment of y.

The ***derivative*** of a function $y = f(x)$ at a point x is the limit of the ratio

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

where $\Delta y = f(x+\Delta x) - f(x)$ is the increment of the function corresponding to the increment of the argument Δx. The derivative y' is also denoted by y'_x, $\dfrac{dy}{dx}$, $f'(x)$, $\dfrac{df}{dx}$.

The operation of finding the derivative y' is usually called ***differentiation of the function***.

Example 11.1 Calculate the derivative of the function $y = x^2$.

Solution. The function $f(x) = x^2$ has the increment

$$\Delta f = f(x+\Delta x) - f(x) = (x+\Delta x)^2 - x^2 = (x+\Delta x - x)(x+\Delta x + x) = \Delta x(2x+\Delta x).$$

Find the limit:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x+\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (2x+\Delta x) = 2x + 0 = 2x$$

.

Thus,

$$y' = 2x.$$

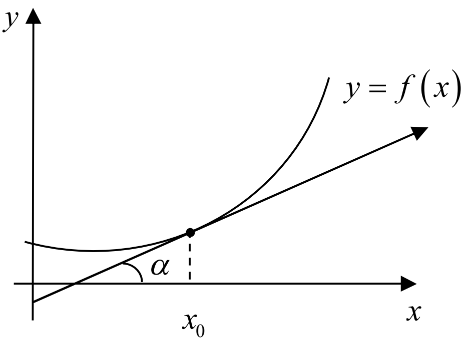


Figure 11.1

Geometrical sense of the derivative. The value of the derivative of the function $f(x)$ at the point x_0 is equal to the tangent of the angle formed by the positive direction of the x-axis and by the positive direction of the tangent line drawn to the graph of this function at the point with the x_0 abscissa (Figure 11.1):

$$tg\alpha = f'(x_0).$$

This quantity is denoted by the term *slope*.

Let $y = f(t)$ be the function describing the path y traversed by a body by the time t. Then the derivative $f'(t)$ is the *velocity* of the body at the instant t (***physical sense of the derivative***).

Basic rules for finding derivatives

Let C be a constant, $u = u(x)$, $v = v(x)$ be functions having derivatives. Then:

1) the constant C can be “moved outside” or “moved through” the derivative:

$$(Cu)' = Cu';$$

2) the addition symbol can be moved through the derivative:

$$(u \pm v)' = u' \pm v';$$

3) for a product of functions: $(uv)' = u'v + uv'$;

Table 11.1 – *Table of derivatives of basic functions*

№	Function	Derivative	№	Function	Derivative
1.	Constant	$C' = 0$	5.	Sine	$(\sin u)' = \cos u \cdot u'$
2.	Power function	$(u^a)' = a u^{a-1} \cdot u'$	6.	Cosine	$(\cos u)' = -\sin u \cdot u'$
2a		$x' = 1$	7.	Tangent	$(\tan u)' = \frac{1}{\cos^2 u} \cdot u'$
2b		$(\sqrt{u})' = \frac{1}{2\sqrt{u}} \cdot u'$	8.	Cotangent	$(\cot u)' = -\frac{1}{\sin^2 u} \cdot u'$
2c		$\left(\frac{1}{u}\right)' = -\frac{1}{u^2} \cdot u'$	9.	Arcsine	$(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u'$
3.	Exponential function	$(a^u)' = a^u \ln a \cdot u'$	10.	Arccosine	$(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u'$
3a	Exponent	$(e^u)' = e^u \cdot u'$	11.	Arc-tangent	$(arctgu)' = \frac{u'}{1+u^2}$
4.	Logarithmic function	$(\log_a u)' = \frac{1}{u \ln a} \cdot u'$	12.	Arc-cotangent	$(arcctgu)' = \frac{u'}{1+u^2}$
4a	Natural logarithm	$(\ln u)' = \frac{1}{u} \cdot u'$			

4) for a quotient (division) of functions: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$;

5) for a composite function: $\left(f\left[u(x)\right]\right)' = f'_u \cdot u'_x$;

6) for the inverse function: $y'_x = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x'_y}$;

Example 11.2 Find the derivatives of the functions:

a) $y = 2x+1$, b) $y = 3x^3+2$, c) $y=\sqrt{x^3}$, d) $y=\sin^2 x$,

e) $y=\cos 3x$, f) $y=\ln\tan x$, g) $y=\ln 2x\cdot\tan x$, h) $y=\dfrac{e^{2x}}{x}$

i) $y=\ln\sin x^2$, j) $y=x^3\cdot\sin 3x$, k) $y=\dfrac{arctg^2 x}{\sqrt{x}}$.

Solution: a) $y'=(2x+1)'=(2x)'+1'=2\cdot x'+0=2\cdot 1=2$;

b) $y'=\left(3x^3+2\right)'=\left(3x^3\right)'+2'=3\cdot\left(x^3\right)'+0=3\cdot 3x^2=9x^2$;

c) $y'=\left(\sqrt{x^3}\right)'=\dfrac{1}{2\sqrt{x^3}}\cdot\left(x^3\right)'=\dfrac{1}{2\sqrt{x^3}}\cdot 3x^2=\dfrac{3x^2}{2\sqrt{x^3}}=\dfrac{3x^2}{2x^{\frac{3}{2}}}=\dfrac{3x^{2-\frac{3}{2}}}{2}=\dfrac{3x^{\frac{1}{2}}}{2}=\dfrac{3\sqrt{x}}{2}$,

$$y'=\left(\sqrt{x^3}\right)'=\left(x^{\frac{3}{2}}\right)'=\frac{3}{2}x^{\frac{1}{2}}=\frac{3\sqrt{x}}{2};$$

d) $y'=\left((\sin x)^2\right)'=2\sin x\cdot(\sin x)'=2\sin x\cdot\cos x=\sin 2x$;

e) $y'=(\cos(3x))'=-\sin(3x)\cdot(3x)'=-\sin 3x\cdot 3\cdot x'=-\sin 3x\cdot 3\cdot 1=-3\sin 3x$;

f) $y'=(\ln(\tan x))'=\dfrac{1}{\tan x}\cdot(\tan x)'=\cot x\cdot\dfrac{1}{\cos^2 x}=\dfrac{\cot x}{\cos^2 x}$;

g) $y'=(\ln 2x\cdot\tan x)'=(\ln 2x)'\cdot\tan x+\ln 2x\cdot(\tan x)'=\dfrac{1}{2x}\cdot(2x)'\cdot\tan x+\ln 2x\cdot\dfrac{1}{\cos^2 x}=$

$$=\frac{1}{2x}\cdot 2\cdot\tan x+\frac{\ln 2x}{\cos^2 x}=\frac{\tan x}{x}+\frac{\ln 2x}{\cos^2 x};$$

h) $y'=\left(\dfrac{e^{2x}}{x}\right)'=\dfrac{\left(e^{2x}\right)'\cdot x-e^{2x}\cdot x'}{x^2}=\dfrac{e^{2x}\cdot(2x)'\cdot x-e^{2x}\cdot 1}{x^2}=\dfrac{e^{2x}\cdot 2\cdot x-e^{2x}}{x^2}=\dfrac{e^{2x}(2x-1)}{x^2}$;

i) $y'=\left(\ln\sin x^2\right)'=\dfrac{1}{\sin x^2}\cdot\left(\sin x^2\right)'=\dfrac{1}{\sin x^2}\cdot\cos x^2\cdot\left(x^2\right)'=$

$$\frac{\cos x^2}{\sin x^2}\cdot 2x=ctgx^2\cdot 2x=2xctgx^2;$$

j) $y'=\left(x^3\cdot\sin 3x\right)'=\left(x^3\right)'\cdot\sin 3x+x^3\cdot\left(\sin 3x\right)'=3x^2\cdot\sin 3x+x^3\cdot\cos 3x\cdot\left(3x\right)'=$

$$=3x^2\sin 3x+x^3\cos 3x\cdot 3=3x^2\sin 3x+$$
$$3x^3\cos 3x=3x^2\left(\sin 3x+x\cos 3x\right);$$

k) $y'=\left(\dfrac{arctg^2x}{\sqrt{x}}\right)'=\dfrac{\left(arctg^2x\right)'\cdot\sqrt{x}-arctg^2x\cdot\left(\sqrt{x}\right)'}{\left(\sqrt{x}\right)^2}=$

$$=\frac{2arctgx\cdot\left(arctgx\right)'\cdot\sqrt{x}-arctg^2x\cdot\frac{1}{2\sqrt{x}}}{x}=\frac{2arctgx\cdot\frac{1}{1+x^2}\cdot\sqrt{x}-arctg^2x\cdot\frac{1}{2\sqrt{x}}}{x}=$$

$$=\frac{\frac{2\sqrt{x}arctgx}{1+x^2}-\frac{arctg^2x}{2\sqrt{x}}}{x}=\frac{arctgx}{x}\cdot\left(\frac{2\sqrt{x}}{1+x^2}-\frac{arctgx}{2\sqrt{x}}\right).$$

If $y=\left(f\left(x\right)\right)^{\varphi(x)}$ is a composite exponential and powerful functions, it is necessary, first of all, take logarithms of the function:

$$\ln y=\varphi\left(x\right)\cdot\ln\left(f\left(x\right)\right),$$

and then to find the derivative of both sides of the equation:

$$\left(\ln y\right)'=\left(\varphi\left(x\right)\cdot\ln\left(f\left(x\right)\right)\right)'.$$

Take into account the following properties of logarithms:

$$\ln\left(a\cdot b\right)=\ln a+\ln b,$$

$$\ln\left(\frac{a}{b}\right)=\ln a-\ln b,$$

$$\ln a^{b}=b\cdot\ln a.$$

Example 11.3 Find the derivative of the function $y=(\cos x)^{x^2}$.

Solution. This function is a composite exponential one, so, it is necessary first to take logarithms of the function:

$$\ln y=\ln(\cos x)^{x^2},\quad \ln y=x^2\ln(\cos x),$$

and then to find the derivative of both sides of the equation:

$$(\ln y)'=\left(x^2\ln(\cos x)\right)',$$

$$\frac{1}{y}y'=\left(x^2\right)'\ln(\cos x)+x^2\left(\ln(\cos x)\right)',$$

$$\frac{1}{y}y'=2x\ln(\cos x)+x^2\frac{-\sin x}{\cos x},$$

$$y'=y\left(2x\ln(\cos x)-x^2\mathrm{tg}x\right),$$

$$y'=(\cos x)^{x^2}\left(2x\ln(\cos x)-x^2\mathrm{tg}x\right).$$

To find the derivative of an implicit function defined by the equation $f(x,y)=0$ it is sufficient:

a) to calculate the derivative, with respect to x, of both sides of the equation, taking y as a function of x;

b) to solve the resulting equation for y'.

Example 11.4 Find the derivative of the function:

$$y=x+\mathrm{arctg}y.$$

Solution: $(y)' = (x + \mathrm{arctg}y)'$, $y' = x' + (\mathrm{arctg}y)'$,

$$y' = 1 + \frac{1}{1+y^2} y',$$

$$y'\left(1+y^2\right) = 1 + y^2 + y',$$

$$y' + y^2 y' = 1 + y^2 + y',$$

$$y^2 y' = 1 + y^2,$$

$$y' = \frac{1+y^2}{y^2}.$$

To find the derivative of a parametrically defined function $x = x(t)$, $y = y(t)$ it is necessary to use the following formula:

$$y'_x = \frac{y'_t}{x'_t}.$$

Example 11.5 Find the derivative of the function:

$$\begin{cases} x = \ln(t+1); \\ y = \sqrt{t+1}. \end{cases}$$

Solution:

$$y'_t = \left(\sqrt{t+1}\right)' = \frac{1}{2\sqrt{t+1}} \cdot (t+1)' = \frac{1}{2\sqrt{t+1}} \cdot 1 = \frac{1}{2\sqrt{t+1}},$$

$$x'_t = \left(\ln(t+1)\right)' = \frac{1}{t+1} \cdot (t+1)' = \frac{1}{t+1} \cdot 1 = \frac{1}{t+1},$$

$$y'_x = \frac{\dfrac{1}{2\sqrt{t+1}}}{\dfrac{1}{t+1}} = \frac{1}{2\sqrt{t+1}} \cdot \frac{t+1}{1} = \frac{t+1}{2\sqrt{t+1}} = \frac{\sqrt{t+1}}{2}.$$

Lecture 12
Derivatives of higher orders. Differentials of functions. L'Hospital rule for evaluating indeterminate forms

The ***second-order derivative*** or the ***second derivative*** of a function $y = f(x)$ is the derivative of the derivative $f'(x)$. The second derivative is denoted by y'', y''_{xx}, $\dfrac{d^2 y}{dx^2}$, $f''(x)$.

The derivative of the second derivative of a function $y = f(x)$ is called the ***third-order derivative***, $y''' = (y'')'$. The ***nth-order derivative*** of the function $y = f(x)$ is defined as the derivative of its $(n-1)\,th$ derivative:

$$y^{(n)} = (y^{(n-1)})'.$$

The *n*th-order derivative is also denoted by $y_x^{(n)}$, $\dfrac{d^n y}{dx^n}$, $f^{(n)}(x)$.

When finding higher order derivatives of an implicit function use the same rules as for the finding the first order derivative of an implicit function.

If the function is parametrically defined, then the derivatives of the second order and above are found by the formulas:

$$y''_{xx} = \frac{\left(y'_x\right)'_t}{x'_t},\quad y'''_{xxx} = \frac{\left(y''_{xx}\right)'_t}{x'_t},\ \ldots,\ y_x^{(n)} = \frac{\left(y_x^{(n-1)}\right)'_t}{x'_t}.$$

Example 12.1 Find the third-order derivative of the function:

$$y = \frac{1}{6}\left(x^2+9\right)\left(x-6\right).$$

Solution:

$$y' = \left(\frac{1}{6}\left(x^2+9\right)\left(x-6\right)\right)' = \frac{1}{6}\cdot\left(\left(x^2+9\right)'\cdot\left(x-6\right)+\left(x^2+9\right)\cdot\left(x-6\right)'\right) =$$

$$=\frac{1}{6}\cdot\left(2x\cdot(x-6)+\left(x^2+9\right)\cdot 1\right)=\frac{1}{6}\cdot\left(2x^2-12x+x^2+9\right)=\frac{1}{6}\left(3x^2-12x+9\right)$$

$$, y''=\left(\frac{1}{6}\left(3x^2-12x+9\right)\right)'=\frac{1}{6}\cdot(6x-12)=\frac{1}{6}\cdot 6\cdot(x-2)=x-2,$$

$$y'''=(x-2)'=1.$$

Example 12.2 Find the second derivative of the function:

$$\begin{cases} x=\tan t; \\ y=\ln(1+\cot t). \end{cases}$$

Solution. This function is parametrically defined one. So, we should use the formulas:

$$y'_x=\frac{y'_t}{x'_t},\quad y''_{xx}=\frac{\left(y'_x\right)'_t}{x'_t}.$$

Let us do it:

$$y'_t=\left(\ln(1+\cot t)\right)'=\frac{1}{1+\cot t}\cdot(1+\cot t)'=$$

$$=\frac{1}{1+\cot t}\cdot\left(-\frac{1}{\sin^2 t}\right)=-\frac{1}{\sin^2 t(1+\cot t)},\ x'_t=(\tan t)'=\frac{1}{\cos^2 t},$$

$$y'_x=\frac{-\dfrac{1}{\sin^2 t\cdot(1+\cot t)}}{\dfrac{1}{\cos^2 t}}=-\frac{\cos^2 t}{\sin^2 t\cdot(1+\cot t)}=-\frac{\cot^2 t}{1+\cot t},$$

$$\left(y'_x\right)'_t=\left(-\frac{\cot^2 t}{1+\cot t}\right)'=-\frac{\left(\cot^2 t\right)'\cdot(1+\cot t)-\cot^2 t\cdot(1+\cot t)'}{(1+\cot t)^2}=$$

$$=-\frac{2\cot t\cdot\left(-\dfrac{1}{\sin^2 t}\right)\cdot(1+\cot t)-\cot^2 t\cdot\left(-\dfrac{1}{\sin^2 t}\right)}{(1+\cot t)^2}=\frac{2\cot t(1+\cot t)-\cot^2 t}{(1+\cot t)^2\sin^2 t}=$$

$$=\frac{2\cot t+2\cot^2 t-\cot^2 t}{\left(1+\cot t\right)^2\sin^2 t}=\frac{\cot t\left(2+\cot t\right)}{\left(1+\cot t\right)^2\sin^2 t},$$

$$y''_{xx}=\frac{\dfrac{\cot t\left(2+\cot t\right)}{\left(1+\cot t\right)^2\sin^2 t}}{\dfrac{1}{\cos^2 t}}=\frac{\cot t\left(2+\cot t\right)\cos^2 t}{\left(1+\cot t\right)^2\sin^2 t}=\frac{\cot^3 t\left(2+\cot t\right)}{\left(1+\cot t\right)^2}.$$

The increment Δx is also called the ***differential of the independent variable*** x and is denoted by dx.

The ***differential*** dy ***of a function*** $y=f(x)$ is the principal part of its increment Δy at the point x. It is equal to the product of its derivative by the differential of the independent variable:

$$dy=y'dx.$$

Basic properties of the differential:

1) $dC=0$, where C is constant,

2) $d\left(Cu\right)=Cdu$,

3) $d\left(u\pm v\right)=du\pm dv$,

4) $d\left(uv\right)=vdu+udv$,

5) $d\left(\dfrac{u}{v}\right)=\dfrac{vdu-udv}{v^2}$ ($v\neq 0$),

6) $df\left(u\right)=f'\left(u\right)du$.

The ***second-order differential*** is the differential of the first-order differential, $d^2y=d(dy)$. If x is the independent variable, then $d^2y=y''dx^2$. In a similar way, one defines differentials of higher orders.

Example 12.3 Find the derivatives and differentials up to the third order inclusive for a function $y=x\cdot\ln x$.

Solution:

$$y'=\left(x\cdot\ln x\right)'=\left(x\right)'\cdot\ln x+x\cdot\left(\ln x\right)'=\ln x+x\cdot\frac{1}{x}=\ln x+1,$$

$$y''=(\ln x+1)'=(\ln x)'+(1)'=\frac{1}{x},\ y'''=\left(\frac{1}{x}\right)'=-\frac{1}{x^2},$$

$$dy=(\ln x+1)dx,\quad d^2y=\frac{1}{x}dx^2,\quad d^3y=-\frac{1}{x^2}dx^3.$$

L'Hospital rule. If $f(x)$ and $g(x)$ are both infinitesimals or both infinites as $x\to a$, that is, if the quotient $\frac{f(x)}{g(x)}$, at $x=a$, is one of the indeterminate forms $\left[\frac{0}{0}\right]$ or $\left[\frac{\infty}{\infty}\right]$, then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

provided that the limit of the ratio of derivatives exists.

The rule is also applicable when $a=\infty$.

If the quotient $\frac{f'(x)}{g'(x)}$ again yields an indeterminate form, at the point $x=a$, of one of the two above-mentioned types and $f'(x)$ and $g'(x)$ satisfy all the requirements that have been stated for $f(x)$ and $g(x)$, we can then pass to the ratio of second derivatives, etc.

Example 12.4 Compute $\lim\limits_{x\to 0}\frac{5x}{\mathrm{arctg}x}$.

Solution:

$$\lim_{x\to 0}\frac{5x}{\mathrm{arctg}x}=\frac{5\cdot 0}{\mathrm{arctg}0}=\left[\frac{0}{0}\right]=$$

Applying the L'Hospital rule we have:

$$=\lim_{x\to 0}\frac{(5x)'}{(\operatorname{arctg}x)'}=\lim_{x\to 0}\frac{5}{\frac{1}{1+x^2}}=\lim_{x\to 0}5\left(1+x^2\right)=5\cdot(1+0)=5\,.$$

Example 12.5 Compute $\lim\limits_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)$.

Solution:

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\frac{1}{\sin^2 0}-\frac{1}{0^2}=\frac{1}{0}-\frac{1}{0}=[\infty-\infty]=$$

Reducing to a common denominator, we get

$$=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\frac{0^2-\sin^2 0}{0^2\cdot\sin^2 0}=\left[\frac{0}{0}\right]=$$

Before applying the L'Hospital rule, we will use one of special limits for trigonometric functions, i.e. $\lim\limits_{\alpha\to 0}\frac{\alpha}{\sin\alpha}=1$:

$$=\lim_{x\to 0}\frac{\left(x^2-\sin^2 x\right)\cdot x^2}{x^2\sin^2 x\cdot x^2}=\lim_{x\to 0}\frac{x^2}{\sin^2 x}\cdot\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\cdot x^2}=1\cdot\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4}=$$

The L'Hospital rule gives

$$=\lim_{x\to 0}\frac{\left(x^2-\sin^2 x\right)'}{\left(x^4\right)'}=\lim_{x\to 0}\frac{2x-2\sin x\cos x}{4x^3}=\lim_{x\to 0}\frac{2x-\sin 2x}{4x^3}=$$

$$=\frac{2\cdot 0-\sin 0}{4\cdot 0^3}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(2x-\sin 2x)'}{\left(4x^3\right)'}=\lim_{x\to 0}\frac{2-2\cos 2x}{12x^2}=$$

$$=\lim_{x\to 0}\frac{1-\cos 2x}{6x^2}=\lim_{x\to 0}\frac{2\sin^2 x}{6x\cdot x}=\lim_{x\to 0}\frac{\sin x\cdot\sin x}{3\cdot x\cdot x}=\frac{1}{3}\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{x}=\frac{1}{3}.$$

To evaluate an indeterminate form like $[0\cdot\infty]$, one should transform the appropriate product $f(x)\cdot g(x)$, into the quotient

$\dfrac{f(x)}{\dfrac{1}{g(x)}}$ or $\dfrac{g(x)}{\dfrac{1}{f(x)}}$ to get one of the indeterminate forms $\left[\dfrac{0}{0}\right]$ or $\left[\dfrac{\infty}{\infty}\right]$, and then to apply the L'Hospital rule.

Example 12.6 Compute $\lim\limits_{x\to 0} x^2 \cdot \ln x$.

Solution:

$$\lim_{x\to 0} x^2 \cdot \ln x = 0^2 \cdot \ln 0 = [0\cdot\infty] = \lim_{x\to 0}\frac{\ln x}{\dfrac{1}{x^2}} = \frac{\ln 0}{\dfrac{1}{0^2}} = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{x\to 0}\frac{(\ln x)'}{\left(x^{-2}\right)'} = \lim_{x\to 0}\frac{\dfrac{1}{x}}{-2x^{-3}} = \lim_{x\to 0}\frac{x^3}{-2x} = \lim_{x\to 0}\frac{x^2}{-2} = \frac{0^2}{-2} = 0.$$

Example 12.7 Compute $\lim\limits_{x\to 0} \sin 6x \cdot \mathrm{ctg}4x$.

Solution:

$$\lim_{x\to 0}\sin 6x\cdot\mathrm{ctg}4x = \sin 0\cdot\mathrm{ctg}0 = [0\cdot\infty] = \lim_{x\to 0}\frac{\sin 6x}{\dfrac{1}{\mathrm{ctg}4x}} = \lim_{x\to 0}\frac{\sin 6x}{\mathrm{tg}4x} = \frac{\sin 0}{\mathrm{tg}0} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to 0}\frac{(\sin 6x)'}{(\mathrm{tg}4x)'} = \lim_{x\to 0}\frac{\cos 6x\cdot 6}{\dfrac{1}{\cos^2 4x}\cdot 4} = \lim_{x\to 0}\frac{6\cos 6x\cos^2 4x}{4} = \frac{6\cdot\cos 0\cdot\cos^2 0}{4} = \frac{6\cdot 1\cdot 1}{4} = \frac{3}{2}.$$

Indeterminate expressions in the form $\left[1^{\infty}\right]$, $\left[\infty^{0}\right]$, $\left[0^{0}\right]$ can be reduced to expressions in the form $\left[\dfrac{0}{0}\right]$ or $\left[\dfrac{\infty}{\infty}\right]$ by taking logarithm and using the formulas $\ln a^b = b\cdot\ln a = \dfrac{\ln a}{1/b}$.

Example 12.8 Compute $\lim\limits_{x\to+\infty}\left(e^x + x\right)^{\frac{1}{x}}$.

Solution:

$$\lim_{x\to+\infty}\left(e^x+x\right)^{\frac{1}{x}}=\left(e^{+\infty}+\infty\right)^{\frac{1}{+\infty}}=\left[\infty^0\right].$$

Taking logarithms and applying the L'Hospital rule, we get

$$\lim_{x\to0}\ln\left(e^x+x\right)^{\frac{1}{x}}=\lim_{x\to0}\frac{1}{x}\cdot\ln\left(e^x+x\right)=\lim_{x\to0}\frac{\ln\left(e^x+x\right)}{x}=\left[\frac{\infty}{\infty}\right]=$$

$$=\lim_{x\to\infty}\frac{\left(\ln\left(e^x+x\right)\right)'}{x'}=\lim_{x\to\infty}\frac{\frac{e^x+1}{e^x+x}}{1}=\lim_{x\to\infty}\frac{e^x+1}{e^x+x}=\frac{e^\infty+1}{e^\infty+\infty}=\left[\frac{\infty}{\infty}\right]=$$

$$=\lim_{x\to\infty}\frac{\left(e^x+1\right)'}{\left(e^x+x\right)'}=\lim_{x\to\infty}\frac{e^x}{e^x+1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\left(e^x\right)'}{\left(e^x+1\right)'}=\lim_{x\to\infty}\frac{e^x}{e^x}=1.$$

Therefore, $\lim\limits_{x\to+\infty}\left(e^x+x\right)^{\frac{1}{x}}=e$.

Example 12.9 Compute $\lim\limits_{x\to0}\left(\cot x\right)^{\frac{1}{\ln x}}$.

Solution:

$$\lim_{x\to0}\left(\cot x\right)^{\frac{1}{\ln x}}=\left(\cot 0\right)^{\frac{1}{\ln 0}}=\left[\infty^0\right].$$

Taking logarithms and applying the L'Hospital rule, we get

$$\lim_{x\to0}\ln\left(\cot x\right)^{\frac{1}{\ln x}}=\lim_{x\to0}\frac{1}{\ln x}\ln\left(\cot x\right)=\lim_{x\to0}\frac{\ln\left(\cot x\right)}{\ln x}=\left[\frac{\infty}{\infty}\right]=$$

$$=\lim_{x\to0}\frac{\left(\ln\left(\cot x\right)\right)'}{\left(\ln x\right)'}=\lim_{x\to0}\frac{\frac{1}{\cot x}\left(\cot x\right)'}{\frac{1}{x}}=\lim_{x\to0}\frac{-\frac{1}{\cot x}\cdot\frac{1}{\sin^2 x}}{\frac{1}{x}}=$$

$$=\lim_{x\to0}\frac{-x}{\cot x\cdot\sin^2 x}=\lim_{x\to0}\frac{-x}{\frac{\cos x}{\sin x}\cdot\sin^2 x}=\lim_{x\to0}\frac{-x}{\cos x\cdot\sin x}=-\lim_{x\to0}\frac{x}{\sin x}\cdot\lim_{x\to0}\frac{1}{\cos x}=1\cdot$$

Note. Of course, we could have solved this limit in another way, using the formula (Appendix C) we could represent $\cos x \cdot \sin x = \frac{1}{2}\sin 2x$, and then we would need to repeat the application of the the L'Hospital rule again. So, it should be remembered that sometimes it is possible to combine the application of the L'Hospital rule and previously known methods in particular the standard limits.

Thus, $\lim\limits_{x\to 0}(\cot x)^{\frac{1}{\ln x}} = e^{-1} = \frac{1}{e}$.

Example 12.10 Compute $\lim\limits_{x\to 1}\left(\frac{x^2+x}{2}\right)^{\frac{1}{x-1}}$.

Solution: $\lim\limits_{x\to 1}\left(\frac{x^2+x}{2}\right)^{\frac{1}{x-1}} = \left(\frac{1^2+1}{2}\right)^{\frac{1}{1-1}} = \left[1^\infty\right]$.

Taking logarithms and applying the L'Hospital rule, we get

$$\lim_{x\to 1}\ln\left(\frac{x^2+x}{2}\right)^{\frac{1}{x-1}} = \lim_{x\to 1}\frac{1}{x-1}\ln\left(\frac{x^2+x}{2}\right) = \lim_{x\to 1}\frac{\ln\left(\frac{x^2+x}{2}\right)}{x-1} = \frac{\ln\left(\frac{1^2+1}{2}\right)}{1-1} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to 1}\frac{\left(\ln\left(\frac{x^2+x}{2}\right)\right)'}{(x-1)'} = \lim_{x\to 1}\frac{\frac{1}{\frac{x^2+x}{2}}\cdot\left(\frac{x^2+x}{2}\right)'}{1} = \lim_{x\to 1}\frac{2}{x^2+x}\cdot\frac{1}{2}(2x+1) = \lim_{x\to 1}\frac{2x+1}{x^2+x} = \frac{2\cdot 1+1}{1^2+1} = \frac{3}{2}.$$

Therefore, $\lim\limits_{x\to 1}\left(\frac{x^2+x}{2}\right)^{\frac{1}{x-1}} = e^{\frac{3}{2}}$.

Lecture 13
Tangent line and normal to function graph.
Application of differentials for approximate calculations.

In your experience with mathematics thus far, you have most likely come across the task of calculating the slope between two points. Whether you know it as "rise over run" or "change in y over change in x", the formula

$$\frac{y_2 - y_1}{x_2 - x_1},$$

you should remember it from our topic about the analytic geometry, will give you the slope between points (x_1, y_1) and (x_2, y_2). For a purpose that will later be revealed, however, what if we were to find the slope of the function at a single point? Such is the nature of the tangent line problem that we are about to explore, and is one of the basic questions of calculus.

The ***equation of the tangent line*** to the graph of a function $y = f(x)$ at a point $M(x_0, y_0)$ has the following form:

$$y - y_0 = y_0'(x - x_0).$$

The *normal* to a curve at point M is a straight line that passes through point M perpendicular to the tangent line.

The ***equation of the normal*** to the graph of a function $y = f(x)$ at a point $M(x_0, y_0)$ has the following form:

$$y - y_0 = -\frac{1}{y_0'}(x - x_0).$$

Example 13.1 Determine the slope of the tangent line to the graph of a function $y = 2tg^2 2x$ at $x = \frac{\pi}{6}$.

Solution. Find the first derivative of $f(x)$ using the Power Rule and Chain Rule, and get

$$y' = 2\left(tg^2 2x\right)' = 4tg2x \cdot \frac{2}{\cos^2 2x}.$$

Plug x value of the indicated point into $f'(x)$ to find the slope at x

$$y'\left(\frac{\pi}{6}\right) = \frac{8tg2\frac{\pi}{6}}{\cos^2 2\frac{\pi}{6}} = \frac{8tg\frac{\pi}{3}}{\cos^2\frac{\pi}{3}} = \frac{8\cdot\sqrt{3}}{\left(\frac{1}{2}\right)^2} = 32\sqrt{3}.$$

The slope of the tangent line to the graph of given function is $32\sqrt{3}$.

Example 13.2 Make the equation of the tangent line and the normal to the curve $y = x^2 - 4x + 3$ at the point with the abscissa $x_0 = 4$.

Solution. Calculate the value of a function at a point $x_0 = 4$:

$$y_0 = 4^2 - 4\cdot 4 + 3 = 16 - 16 + 3 = 3.$$

Find the derivative of the function:

$$y' = \left(x^2 - 4x + 3\right)' = 2x - 4.$$

Calculate the value of the derivative at the point $x_0 = 4$:

$$y_0' = 2\cdot 4 - 4 = 8 - 4 = 4.$$

Then the tangent line (Fig. 11.1) equation has the form:

$$y - y_0 = y_0'\left(x - x_0\right),$$

$$y - 3 = 4\left(x - 4\right), \quad y = 4x - 16 + 3,$$

$$y = 4x - 13,$$

and the equation of the normal (Figure 12.1) is:

$$y - y_0 = -\frac{1}{y_0'}\left(x - x_0\right)$$

$$y-3=-\frac{1}{4}(x-4), \quad y=-\frac{1}{4}x+1+3,$$

$$y=-\frac{1}{4}x+4.$$

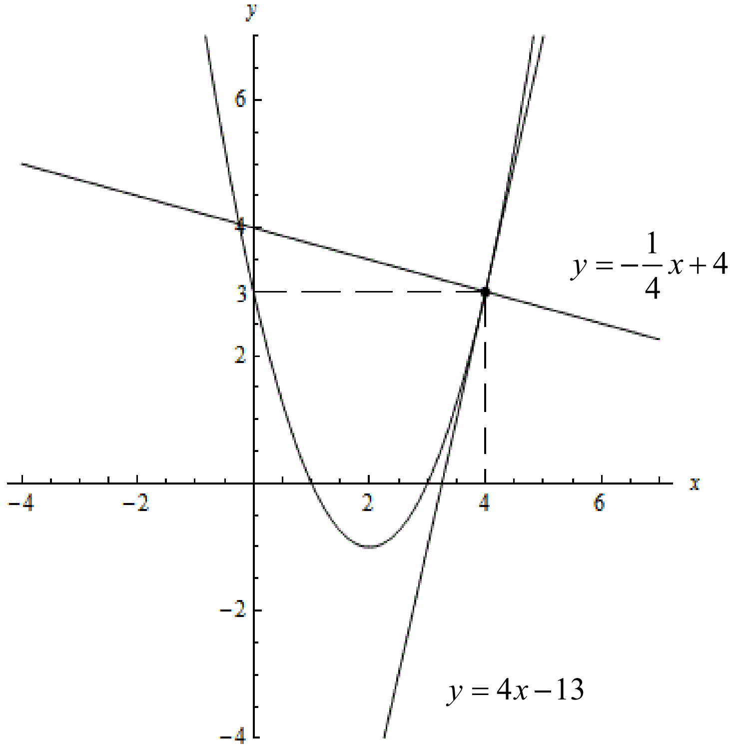


Figure 13.1

Approximate equality $\Delta y \approx dy$ allows the use of the differential for approximate calculations of the function values.

Since

$$dy \approx \Delta y = f(x_0+\Delta x)-f(x_0) \text{ and } dy = f'(x_0)dx \approx f'(x_0)\Delta x,$$

then we have

$$f(x_0+\Delta x)-f(x_0) \approx f'(x_0)\Delta x \quad \text{or} \quad f(x_0+\Delta x) \approx f(x_0)+f'(x_0)\Delta x.$$

Example 13.3 Find the approximate values of the functions using the differentials of the specified functions.

$$\text{a) } 1{,}03^5, \quad \text{b) } \tan 44°.$$

Solution:

a) since $f(x) = x^5$, then $f'(x) = 5x^4$.

From the conditions of the task we have

$x_0 = 1$, $\quad x_0 + \Delta x = 1{,}03$, $\quad \Delta x = 1{,}03 - x_0 = 1{,}03 - 1 = 0{,}03$.

Calculate the value of a function and its derivative at a point $x_0 = 1$:

$$f(1) = 1^5 = 1, \quad f'(1) = 5 \cdot 1^4 = 5.$$

Thus,

$$1{,}03^5 \approx 1 + 5 \cdot 0{,}03 = 1 + 0{,}15 = 1{,}15;$$

b) since $f(x) = \tan x$, then $f'(x) = \dfrac{1}{\cos^2 x}$.

Taking into account that $\tan 44° = \tan(45° - 1°)$, let us take $x_0 = 45°$, then $\Delta x = -1°$, i.e. $x_0 = \dfrac{\pi}{4}$, $\Delta x = -\dfrac{\pi}{180}$. Calculate the value of a function and its derivative at a point $x_0 = \dfrac{\pi}{4}$:

$$f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1, \quad f'\left(\frac{\pi}{4}\right) = \frac{1}{\cos^2\frac{\pi}{4}} = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{1}{\frac{2}{4}} = \frac{1}{\frac{1}{2}} = 2 \cdot$$

Thus,

$$\tan 44° \approx 1 + 2 \cdot \left(-\frac{\pi}{180}\right) = 1 - 2 \cdot \frac{3{,}14}{180} = 1 - \frac{3{,}14}{90} = 0{,}965 \cdot$$

Lecture 14
Conditions for increasing and decreasing the function. Extrema

The function $y = f(x)$ is called ***increasing*** (***decreasing***) on some interval if, for any points x_1 and x_2 which belong to this interval, from the inequality $x_2 > x_1$ we get the inequality $f(x_2) > f(x_1)$ ($f(x_2) < f(x_1)$). If $f(x)$ is continuous on the interval $[a,b]$ and $f'(x) > 0$ ($f'(x) < 0$) for $a < x < b$, then $f(x)$ increases (decreases) on the interval $[a,b]$.

The domain of definition of $f(x)$ may be subdivided into a finite number of intervals of increase and decrease of the function (*intervals of monotonicity*). These intervals are bounded by *critical points* x (where $f'(x) = 0$ or $f'(x)$ does not exist).

A ***maximum*** value of a function is a value greater than those immediately preceding or immediately following. A ***minimum*** value of a function is a value less than those immediately preceding or immediately following.

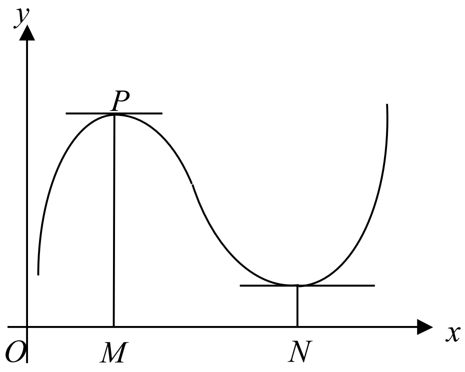


Figure 14.1

If the function is represented by the curve $y = f(x)$ (Figure 14.1), then PM represents a maximum value of y or of $f(x)$, and QN represents a minimum value.

At both P and Q the tangent line is parallel to x-axis, and therefore we have for both maxima and minima,

$$y' = 0 \quad \text{or} \quad f'(x) = 0.$$

The minimum point or maximum point of a function is its *extremal point*, and the minimum or maximum of a function is called the *extremum* of the function. If x_0 is an extremal point of the function $f(x)$, then $f'(x_0)=0$, or $f'(x_0)$ does not exist (*necessary condition for the existence of an extremum*).

When y' changes from $+$ to $-$ at critical point, this point is a maximum, and when y' changes from $-$ to $+$, it is a minimum (*sufficient condition for the existence of an extremum*).

Example 14.1 Find the extrema of the function

$$y=\frac{x}{1+x^2}.$$

Solution. The domain of function definition is:

$$1+x^2\neq 0,\ x\in R.$$

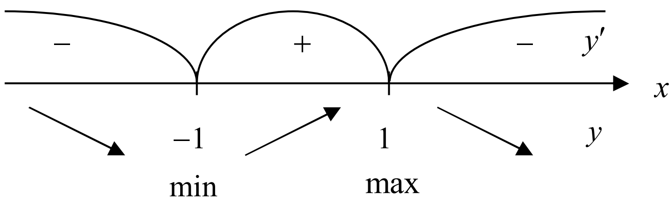
Find the derivative of the given function:

$$y'=\left(\frac{x}{1+x^2}\right)'=\frac{x'\cdot\left(1+x^2\right)-x\cdot\left(1+x^2\right)'}{\left(1+x^2\right)^2}=\frac{1\cdot\left(1+x^2\right)-x\cdot 2x}{\left(1+x^2\right)^2}=\frac{1+x^2-2x^2}{\left(1+x^2\right)^2}=\frac{1-x^2}{\left(1+x^2\right)^2}.$$

Solve the equation $y'=0$:

$$\frac{1-x^2}{\left(1+x^2\right)^2}=0,\ 1-x^2=0,\quad x^2=1,\quad x=\pm 1.$$

Put all critical point taking into account the point where our function does not exist (or undetermined) on the axis and investigate the sign at the obtained intervals.



To determine what the sign of y' is in the interval $(-\infty;-1)$, it is sufficient to determine the sign of y' at some point of the interval. For example, taking $x=-2$, we get

$$y'(-2)=\frac{-2}{1+(-2)^2}=\frac{-2}{1+4}=-\frac{2}{5},$$

hence, $y'<0$ in the interval $(-\infty;-1)$ and the function in this interval decreases.

Therefore, the function increases in the interval $x\in(-1,1)$, decreases in the interval $x\in(-\infty,-1)\cup(1,+\infty)$, $x=-1$ is the minimum point of the function, $x=1$ is the maximum one.

$$y_{\min}(-1)=\frac{-1}{1+(-1)^2}=-0,5, \quad y_{\max}(1)=\frac{1}{1+1^2}=0,5,$$

$A(-1;-0,5)$, $B(1;0,5)$ are the extremal points.

To find the ***largest*** (***smallest***) ***value*** of a function $f(x)$ ***on a segment*** $[a,b]$, it is necessary to choose the largest (smallest) values of the function from the values of the function on the boundaries of the segment and at the critical points belonging to this segment.

Example 14.2 Find the largest and smallest values of a function $y=x^3+9x^2-1$ on a segment $[-2,2]$.

Solution. Since

$$y'=\left(x^3+9x^2-1\right)'=3x^2+18x,$$

it follows that the critical points of the function are

$$3x^2+18x=0,$$

$$3x\cdot(x+6)=0,$$

$$x=0\in[-2;2], \quad x=-6\notin[-2;2].$$

Comparing the values of the function at $x=0$ and at the end-points of the given interval

$$y(0)=0^3+9\cdot 0^2-1=-1,$$

$$y(-2)=(-2)^3+9\cdot(-2)^2-1=-8+36-1=27,$$

$$y(2)=2^3+9\cdot 2^2-1=8+36-1=43,$$

we conclude that the function attains its smallest value $m=-1$ at the point $x=0$ and the greatest value $M=43$ at the point $x=2$:

$$\max_{[-2;2]} y=y(2)=43, \quad \min_{[-2;2]} y=y(0)=-1.$$

Example 14.3 Find the maximum value and the minimum value attained by $f(x)=\dfrac{1}{x(1-x)}$ in the interval $[2,3]$.

Solution. Note that the domain of $f(x)$ does not contain $x=0$ and $x=1$, and these points are not in the interval $[2,3]$. Find critical points. Compute

$$f'(x)=-\frac{1-2x}{x^2(1-x)^2}, \; f'(x)=0, \; -\frac{1-2x}{x^2(1-x)^2}=0, \; 2x=1, \; x=\frac{1}{2}.$$

Therefore, the only possible critical point is $x=\dfrac{1}{2}$. As this point is not in the interval $[2,3]$, it is not a critical point. Compute $f(x)$ only at the boundaries of the closed interval

$$f(3)=\frac{1}{3(1-3)}=-\frac{1}{6}, \; f(2)=\frac{1}{2(1-2)}=-\frac{1}{2}.$$

Compare the data resulted in Step 2 to make conclusions: $f(x)$ attains its absolute maximum value $f(3)=-\dfrac{1}{6}$ at $x=3$ and f(x) attains its absolute minimum value $f(2)=-\dfrac{1}{2}$ at $x=2$.

Lecture 15
Concavity and inflection points. Asymptotes. General scheme of analysis of a function

A curve is said to be ***concave upwards*** (look at Figure 15.1) (***concave downwards*** (look at Figure 15.2)) at a point M , when in the immediate neighborhood of M it lies wholly above (below) the tangent at M .

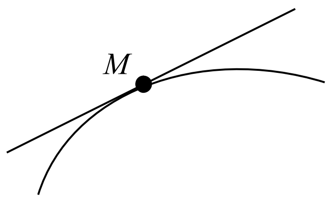


Figure 15.1

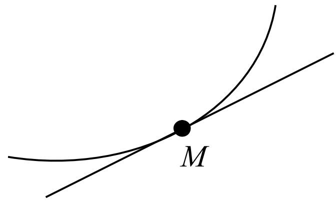


Figure 15.2

A sufficient condition for the concavity upwards (downwards) of a graph $y = f(x)$ is that the following inequality be fulfilled in the appropriate interval:

$$f''(x) < 0 \;\; (f''(x) > 0).$$

A ***point of inflexion*** is a point M where $f''(x)$ changes sign, the curve being concave upwards on one side of this point, and concave downwards on the other.

We can identify such points by first finding where $f''(x) = 0$ and then checking to see whether it changes sign from positive to negative or negative to positive at these points.

Example 15.1 Find the intervals of concavity and the inflection points of the graph of the functions:

a) $y = x^3 - 15x^2 + 36x + 2$; b) $y = x \cdot e^x$.

Solution: a) the domain of the function definition is: $x \in R$.

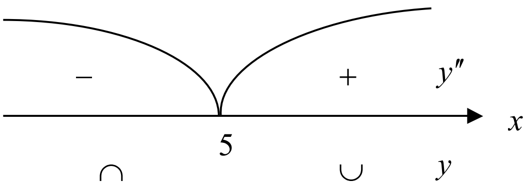
Find the derivative of the given function:

$$y' = (x^3 - 15x^2 + 36x + 2)' = 3x^2 - 30x + 36,$$

$$y'' = (3x^2 - 30x + 36)' = 6x - 30.$$

Solve the equation $y'' = 0$:

$$6x - 30 = 0, \quad x = 5.$$



Thus, when passing through a point $x = 5$, the second derivative changes its sign: in the interval $(-\infty; 5)$ it is negative and the curve is concave up, and in the interval $(5; +\infty)$ it is positive and the curve is concave down. Therefore, $x = 5$ is the inflection point:

$$y(5) = 5^3 - 15 \cdot 5^2 + 36 \cdot 5 + 2 = 125 - 375 + 180 + 2 = -68;$$

b) the domain of the function definition is: $x \in R$.

Find the second order derivative of the given function:

$$y' = \left(x \cdot e^x\right)' = x' \cdot e^x + x \cdot \left(e^x\right)' = e^x + x \cdot e^x = e^x\left(1 + x\right),$$

$$y'' = \left(e^x \cdot (1+x)\right)' = \left(e^x\right)' \cdot (1+x) + e^x \cdot (1+x)' = e^x \cdot (1+x) + e^x \cdot 1 = e^x\left(1+x+1\right) = e^x\left(2+x\right).$$

Solve the equation $y'' = 0$:

$$e^x\left(2 + x\right) = 0, \quad 2 + x = 0, \quad x = -2.$$

	$(-\infty; -2)$	-2	$(-2; +\infty)$
$f''(x)$	$-$	0	$+$
$f(x)$	$\cap$	$-0,27$	$\cup$

Thus, when passing through a point $x=-2$, the second derivative changes its sign: in the interval $(-\infty;-2)$ it is negative and the curve is concave up, and in the interval $(-2;+\infty)$ it is positive and the curve is concave down. Therefore, $x=-2$ is the inflection point:

$$y(-2)=-2\cdot e^{-2}=-\frac{2}{e^2}\approx-0,27\,.$$

If a point (x,y) is in continuous motion along a curve $y=f\left(x\right)$ in such a way that at least one of its coordinates approaches infinity (and at the same time the distance of the point from some straight line tends to zero), then this straight line is called an ***asymptote*** of the curve.

The curve $y=f\left(x\right)$ has a *vertical asymptote* $x=a$ if $f\left(x\right)\to\pm\infty$ as $x\to a\pm 0$:

$$\lim_{x\to a\pm 0} f\left(x\right)=\pm\infty\,.$$

To determine the vertical asymptotes, it is necessary to find those values of the argument, near which $f\left(x\right)$ increases without limit in absolute magnitude.

To determine an *oblique asymptote* $y=kx+b$ of the curve $y=f\left(x\right)$, it is necessary to find the numbers k and b from the formulas

$$k=\lim_{x\to\pm\infty}\frac{f\left(x\right)}{x},\quad b=\lim_{x\to\pm\infty}\left(f\left(x\right)-kx\right).$$

If $k=\pm\infty$, there are no oblique asymptotes.

If $k=0$, then an oblique asymptote turns into a *horizontal* one $y=b$:

$$b=\lim_{x\to\pm\infty} f\left(x\right).$$

Example 15.2 Find the asymptotes of the function:

$$y=\frac{x^2-3x+3}{x-1}.$$

Solution. The domain of the function definition is:

$$x-1\neq 0,\ x\neq 1,\ x\in(-\infty,1)\cup(1,+\infty).$$

Therefore, $x=1$ is a vertical asymptote of the given function:

$$\lim_{x\to 1-0}\frac{x^2-3x+3}{x-1}=\frac{(1-0)^2-3\cdot(1-0)+3}{1-0-1}=\frac{1-3+3}{-0}=\frac{1}{-0}=-\infty,$$

$$\lim_{x\to 1+0}\frac{x^2-3x+3}{x-1}=\frac{(1+0)^2-3\cdot(1+0)+3}{1+0-1}=\frac{1-3+3}{+0}=\frac{1}{+0}=+\infty.$$

Find the coefficients k and b for this function:

$$k=\lim_{x\to\infty}\frac{\frac{x^2-3x+3}{x-1}}{x}=\lim_{x\to\infty}\frac{x^2-3x+3}{x(x-1)}=\lim_{x\to\infty}\frac{x^2-3x+3}{x^2-x}=\frac{\infty^2-3\cdot\infty+3}{\infty^2-\infty}=\left[\frac{\infty}{\infty}\right]=$$

$$=\lim_{x\to\infty}\frac{(x^2-3x+3)'}{(x^2-x)'}=\lim_{x\to\infty}\frac{2x-3}{2x-1}=\frac{2\cdot\infty-3}{2\cdot\infty-1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(2x-3)'}{(2x-1)'}=\frac{2}{2}=1$$

,

$$b=\lim_{x\to\infty}\left(\frac{x^2-3x+3}{x-1}-x\right)=\lim_{x\to\infty}\frac{x^2-3x+3-x(x-1)}{x-1}=\lim_{x\to\infty}\frac{x^2-3x+3-x^2+x}{x-1}=$$

$$=\lim_{x\to\infty}\frac{-2x+3}{x-1}=\lim_{x\to\infty}\frac{-2x+3}{x-1}=\frac{-2\cdot\infty+3}{\infty-1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(-2x+3)'}{(x-1)'}=\frac{-2}{1}=-2,$$

so $y=x-2$ is an oblique asymptote of the given function.

General scheme of analysis of a function

1. Determine the domain in which the function is defined.
2. Find the points at which the graph crosses the coordinate axes. Notice that the y*-intercept* occurs where $x=0$, and the x*-intercept* occurs where $y=0$.

3. Determine whether the function is odd or even and whether it is periodic.

A function is *even* if $y(-x)=y(x)$. The graph of such function is symmetric with respect to the y-axis. A function is *odd* if $y(-x)=-y(x)$. The graph of such function is symmetric with respect to the origin. A function is said to be *periodic* if $y(x+T)=y(x)$ for some *nonzero* constant T (a *period* of the function).

4. Find extremal points and intervals of monotonicity.

5. Determine the directions of convexity of the graph and its inflection points.

6. Find the asymptotes of the graph.

7. Draw the graph.

Example 15.3 Examine the function $y=\dfrac{x^2}{x-1}$ and construct its graph:

Solution:

1. The domain of the function definition is:

$$x-1\neq 0,\ x\neq 1,\ x\in(-\infty;1)\cup(1,+\infty).$$

2. Points of intersection of a graph with coordinate axes:

$$Ox:\ y=0,\ \frac{x^2}{x-1}=0,\ x^2=0,\ x=0,\ O(0,0).$$

$$Oy:\ x=0,\ y=\frac{0^2}{0-1}=0,\ O(0,0).$$

3. This function is neither odd nor even, since

$$y(-x)=\frac{(-x)^2}{-x-1}=\frac{x^2}{-x-1}=-\frac{x^2}{x+1}\neq \pm y(x).$$

Obviously, this function is nonperiodic.

4. Find the derivative of the function:

$$y' = \left(\frac{x^2}{x-1}\right)' = \frac{(x^2)' \cdot (x-1) - x^2 \cdot (x-1)'}{(x-1)^2} = \frac{2x \cdot (x-1) - x^2 \cdot 1}{(x-1)^2} =$$

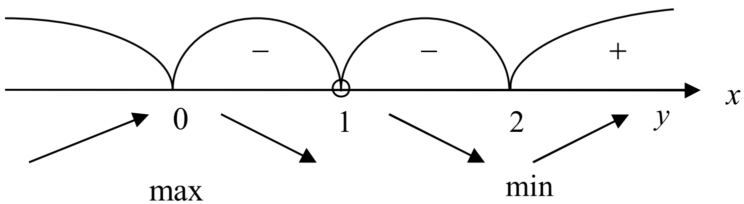
$$= \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}.$$

Solve the equation $y' = 0$:

$$\frac{x(x-2)}{(x-1)^2} = 0,$$

$$x(x-2) = 0,$$

$$x = 0, \quad x = 2.$$



Therefore, the function increases in the interval $x \in (-\infty, 0) \cup (2, +\infty)$, decreases in the interval $x \in (0,1) \cup (1,2)$,

$$x_{\max} = 0, \quad y_{\max}(0) = \frac{0^2}{0-1} = 0,$$

$$x_{\min} = 2, \quad y_{\min}(2) = \frac{2^2}{2-1} = \frac{4}{1} = 4,$$

$O(0;0)$ is the graph point corresponding to the maximum,
$M_1(2;4)$ is the graph point corresponding to the minimum.

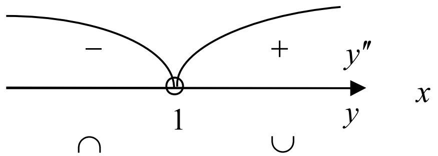
5. Find the second derivative of the function:

$$y''=\left(\frac{x^2-2x}{(x-1)^2}\right)'=\frac{(x^2-2x)'\cdot(x-1)^2-(x^2-2x)\cdot((x-1)^2)'}{(x-1)^4}=$$

$$=\frac{(2x-2)\cdot(x-1)^2-(x^2-2x)\cdot 2(x-1)}{(x-1)^4}=\frac{2(x-1)\cdot(x-1)^2-(x^2-2x)\cdot 2(x-1)}{(x-1)^4}=$$

$$=\frac{2(x-1)\cdot((x-1)^2-(x^2-2x))}{(x-1)^4}=\frac{2(x^2-2x+1-x^2+2x)}{(x-1)^3}=\frac{2}{(x-1)^3},$$

$\frac{2}{(x-1)^3}\neq 0$, there are no inflection points.



In the interval $x\in(-\infty;1)$ $y''<0$ and the curve is concave up, and in the interval $x\in(1;+\infty)$ $y''>0$ and the curve is concave down.

6. $x=1$ is a vertical asymptote, since

$$\lim_{x\to 1-0}\frac{x^2}{x-1}=\frac{(1-0)^2}{1-0-1}=\frac{1}{-0}=-\infty,\quad \lim_{x\to 1+0}\frac{x^2}{x-1}=\frac{(1+0)^2}{1+0-1}=\frac{1}{+0}=+\infty.$$

Find the coefficients k and b for this function:

$$k=\lim_{x\to\infty}\frac{\frac{x^2}{x-1}}{x}=\lim_{x\to\infty}\frac{x}{x-1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{x'}{(x-1)'}=1,$$

$$b=\lim_{x\to\infty}\left(\frac{x^2}{x-1}-x\right)=\lim_{x\to\infty}\frac{x^2-x\cdot(x-1)}{x-1}=\lim_{x\to\infty}\frac{x^2-x^2+x}{x-1}=\lim_{x\to\infty}\frac{x}{x-1}=1$$

,

$y=x+1$ an oblique asymptote.

7. Draw the graph (Figure 15.3).

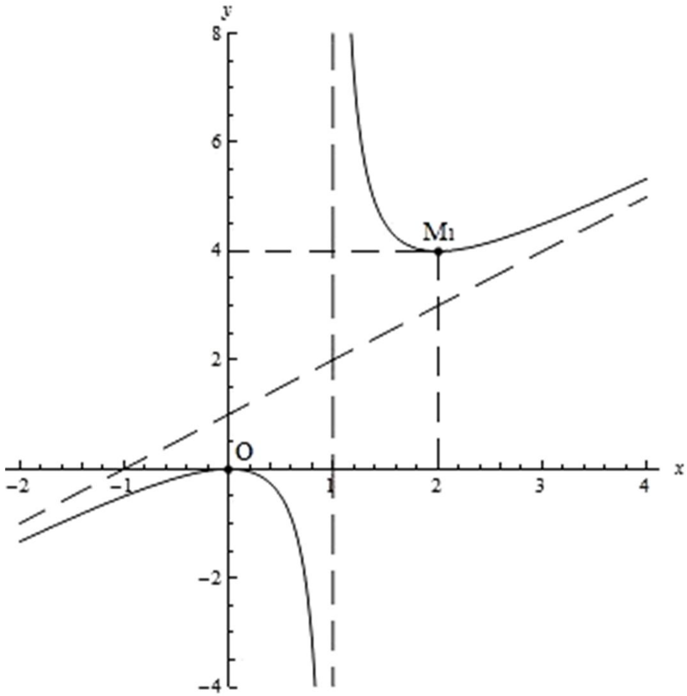


Figure 15.3

QUESTIONS TO CONSOLIDATE LECTURE

Lecture 1

1. What is a determinant?
2. What is a minor?
3. What is a cofactor of a determinant element?
4. By what rule the value of the determinant of the n-th order is calculated?
5. Formulate the rules of the “cross” and “triangles” for calculating respectively the determinants of the second and third order
6. What are the basic properties of the determinant? Which of them can we use to calculate it.
7. How a determinant of a triangular form is calculated?
8. Will be the value of the determinant changed if the elements of some column are multiplied by 5? If so, how much?
9. Which of the properties can we use to simplify calculation the determinant of any order? Explain your answer.

Lecture 2

1. What is a matrix?
2. Which of matrix is called non-degenerate?
3. How are doing the operations of adding (subtracting) matrices and multiplying the matrix by the number?
4. What is the difference between multiplication of the matrix by the scalar and the multiplication of the determinant by the number?
5. How is operation of multiplication of the matrixes carried out? What are the properties of this operation?
6. Which of matrix has determinant?
7. What is an inverse matrix and how is it calculated?
8. Does any matrix have an inverse matrix? Why?
9. How can you check the accuracy of the found inverse matrix?

Lecture 3

1. What kind of the form has a system of m linear algebraic equations (SLAE) with n unknowns?
2. Which of a system is called the compatible?
3. Which of a system is called the defined system?
4. What is the system of linear algebraic equations which has all free terms are zeros? Does such system have a solution? How many?
5. How can we find a solution of a square SLR with an inverse matrix?
6. How should you do the check out of your solution? What result can you get?
7. How to solve the square system of linear equations by Cramer's rule?
9. Could the inverse matrix method and Cramer method be applied to solve any kind of systems? Explain your answer.
10. How is an arbitrary SLAE solved by the Gaussian elimination method?
11. Is it possible to determine the consistency of the system using the Gaussian elimination method? Explain your answer.
12. How can we know by performing the Gaussian elimination method that the system does not have a solution?

Lecture 4

1. What is a vector?
2. What are the direction cosines?
3. Formulate properties of a scalar product.
4. What do determine the sign of a scalar product?
5. How to use the concept of scalar product in mechanics?
6. What is a scalar square?
7. Can we use such transformations to the vector product?
8. What is a right-hand system?
9. Do you need to consider the orientation of a system?
10. What is a geometric interpretation of the vector product?
11. How to calculate the triangular area?
12. Could items order be changed in the vector product?

13. What can you tell about coordinates of collinear vectors?

Lecture 5

1. What is a tensor?
2. Is it important to understand a sense of tensors? Could an index position change the sense of tensors or their properties?
3. When do we say that it is a convolution? How can be it found?
4. What is contravariance?
5. What is Einstein's rule?
6. What is a tensor invariant? Have all tensors an invariant?
7. What is a tensor trace?
8. What is a metric tensor? What is its properties?
9. What is Kronecker symbol?

Lecture 6

1. What is a general equation of a straight line on a plane?
2. What do know special cases of the general equation of a straight line?
3. How to write the equation of a straight line?
4. How to find the slope of a line?
5. What is happened with line if its slope is zero?
6. What can you say about the parallel lines? perpendicular lines?
7. Do we have deference between concepts as distance between two points and distance between point and lines?
8. How to find the coordinates of the lines intersection point?

Lecture 7

1. What is a normal vector?
2. Tell all special case of a plane general equation.
3. What kind of plane equations do you know?
4. What are relationships between two planes in space have?
5.
6. What is the condition of perpendicularity of two planes?
7. What are the differences between a normal vector and a direction vector?

8. Can we determine the coordinates of a direction vector knowing the equation of a line?

9. Can we transform the canonic equation of a line in other forms of equations? How could we do it?

10. Give some example about the location of lines in a space, and location of planes in a space, location of a line and a plane in a space.

Lecture 8

1. What is a circle?
2. What is its standard equation of a circle?
3. What is an ellipse?
4. What are the foci of the ellipse and where are they located?
5. What does the eccentricity of the ellipse characterize ?
6. What properties of the ellipse could we learn from its canonical equation?
7. What is a hyperbola?
8. What features do have a hyperbola?
9. What are hyperbola asymptotes?
10. What is a parabola?
11. Give some examples of special case of its graphs?
12. What are polar coordinates?
13. What are the relationships between polar and Cartesian coordinates?

Lecture 9

1. What is a sphere? What is its equation?
2. What can we get in the cross section of the ellipsoid?
3. What is a cone?
4. What is a canonical equation of an elliptical cone of the second order?
5. Do all surfaces have similarities? Which of them have?
6. What are differences between a single-cavity hyperboloid and a double-cavity hyperboloid?
7. What is a hyperbolic paraboloid? How could it get?
8. How could we get a cylindrical surface?

9. What is type of curve obtained in the cross section of the cylinder with a plane perpendicular to the generating?

Lecture 10

1. Explain how do you understand this “limit of the function $f(x)$ from the left and the limit of the function $f(x)$ from the right at the point a”? Is it enough to assert that a function have the limit at the point a ?
2. What is an infinitesimal function?
3. Call some properties of infinitesimal and infinitude functions.
4. Which of fundamental limits do know?
5. What kinds of indeterminate forms do you know? Tell us the features of disclosing some of them.

Lecture 11

1. What is an increment of a variable?
2. What is a derivative of a function?
3. How to find the derivative of a compose function?
4. In which case should we use the Chain Rule?
5. How to find the derivative of an implicit function?
6. What is a logarithmic differentiation? When could we use it?

Lecture 12

1. What is the second-order differential? How can we find it?
2. Explaine the L’Hospital rule.
3. Can we use the L’Hospital rule at all examples or not?
4. How can we use this rule if you need to compute the limit having one of these indeterminate forms $\left[1^{\infty}\right]$, $\left[\infty^{0}\right]$, $\left[0^{0}\right]$?
5. Can you combine the using of the L’Hospital rule with other ways or previously learned technology?

Lecture 13

1. What is a slope of a curve?
2. What is an equation of the tangent line?
3. What is an equation of the normal line?

4. How can we the concept of a function derivative in a approximation calculus?

5. Explain why we can do it.

Lecture 14

1. What are conditions of increasing and decreasing the function?
2. What are properties of a critical point?
3. How can be found critical points?
4. What is a necessary condition for the existence of an extremum?
5. What are the exteremal points
6. How can we find the largest (smallest) value of a function $f(x)$ on a segment $[a,b]$?

Lecture 15

1. What is a point of inflexion? How to find it?
2. What does it mean when we say that the curve is concave upwards?
3. What is a sufficient condition for the concavity upwards (downwards) of a function graph?
4. How can we determine that this cure is concave upwards?
5. What is an asymptote? What kind of them do you know?
6. What should we do to find an asymptote to the graph of a function?
7. What is a general scheme of a function analysis?
8. Should we calculate the arbitrary points to draw a graph of a function?

LIST of USEFUL RECOURSES

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APPENDICES

APPENDIX A

The cross-sectional method

To find out the shape of a surface in a space according to its equation

$$\varphi(x, y, z) = 0 \qquad \text{(A.1)}$$

the so-called ***section method*** is often used. It consists of analyzing the intersections of a surface with planes parallel to the coordinate planes, for example with planes like $z = c$, c where the parameter c runs through all real values. The equations system

$$\begin{cases} \varphi(x, y, z) = 0 \\ z = c \end{cases} \qquad \text{(A.2)}$$

gives the corresponding intersection for each value c. The criterion for a point $M(x, y, z)$ to belong to this intersection is the following conditions: a) $z = c$; b) the coordinates x and y are its projections onto the coordinate plane xOy, i.e. coordinates of a point $N(x, y, 0)$ satisfy to the equation

$$\varphi(x, y, c) = 0 \qquad \text{(A.3)}$$

Knowing these intersections, i.e. a curves equation (A.3), we can imagine the shape of the surface. Note that the indicated "X-ray" of the surface can be carried out by other planes, but they must be parallel to each other.

Usually, when studying the shape of a surface by the method of sections, two points of view on the equation (A.3). The first is that it is interpreted as the equation of projection onto the coordinate plane xOy by the sections (A.2). According to the second point of view, it is assumed that in the secant plane there is a rectangular coordinate system with the origin at the point O' intersection of the secant plane with the axis Oz and axes, $O'x$ and

$O'y$, which are projected onto the corresponding axes Ox and Oy of a system coordinates $Oxyz$. This allows us to speak of equation (A.3) as the equation of the section (A.2) in the secant plane.

Draw the surface $z = x^2 + y^2$.

Fix a value of x as $x = 0$ and draw a parabola $z = y^2$ in a plane zOy. Fix $y = 0$ and draw a parabola $z = x^2$ in a plane zOx. If we assume that $z = 0$, then we get the origin point $O(0,0,0)$. Let it be $z = 4$ then we will find a section of an elliptic paraboloid by this plane

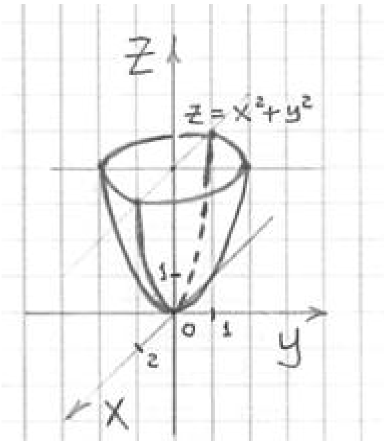


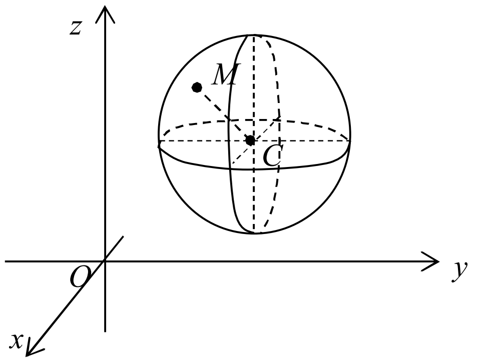
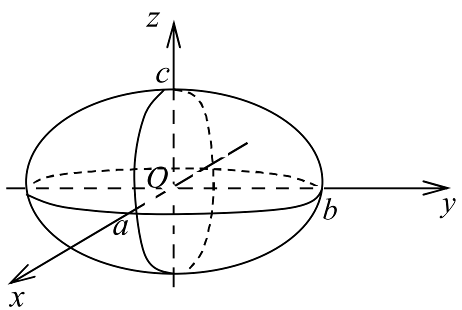
Figure A.1

$$\begin{cases} z = 4, \\ x^2 + y^2 = 4; \end{cases} \Rightarrow x^2 + y^2 = 4 \ ,$$

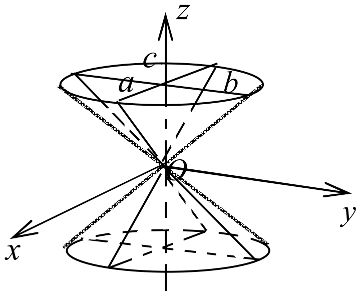
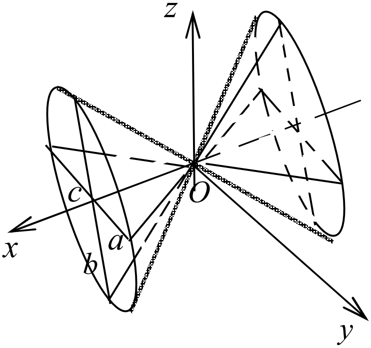
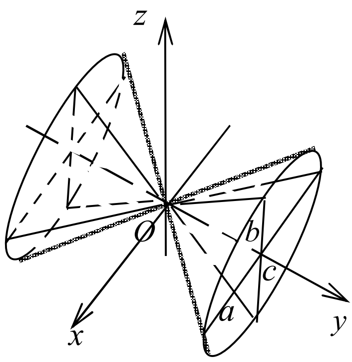
$x^2 + y^2 = 4$ is a circle centered at origin point and it has radius $R = 2$. Draw it. We get a desired surface at the figure A.1.

APPENDIX B

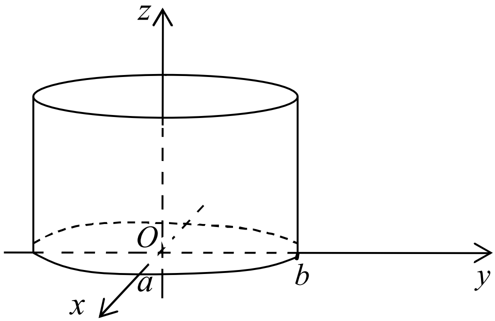
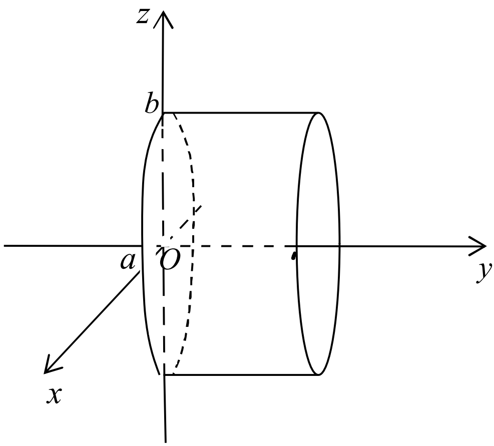
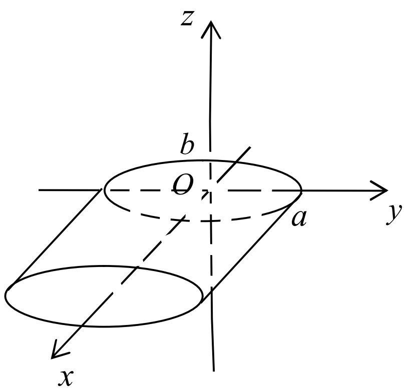
Table B.1 –Graphs and equations of surfaces of the second order

Name	Equation	Graph
Sphere	$(x-x_0)^2 +$ $+(y-y_0)^2 +$ $+(z-z_0)^2 =$ $= R^2$	
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	

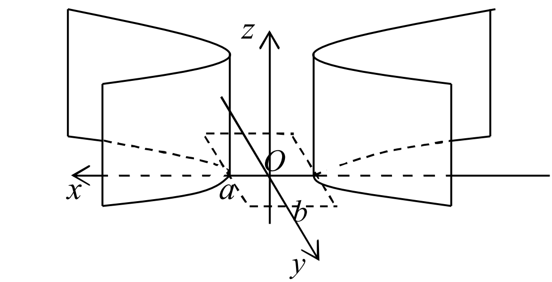
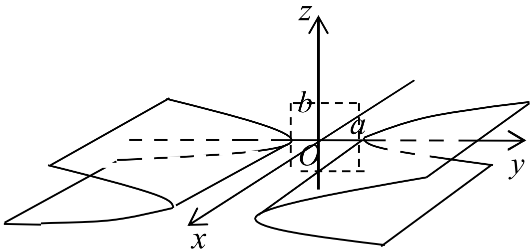
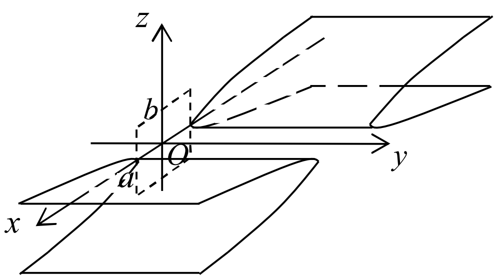
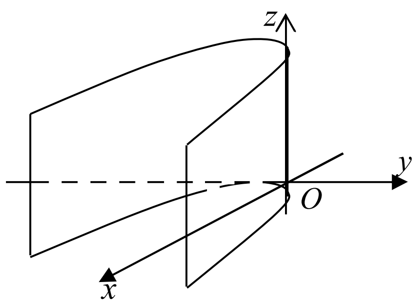
Continued Table B.1

Cone	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=$	
	$\frac{y^2}{a^2}+\frac{z^2}{b^2}-\frac{x^2}{c^2}=$	
	$\frac{x^2}{a^2}+\frac{z^2}{b^2}-\frac{y^2}{c^2}=$	

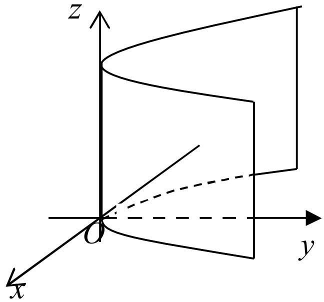
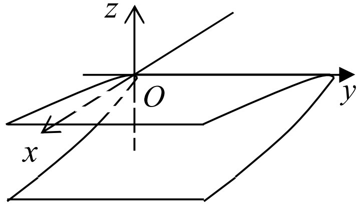
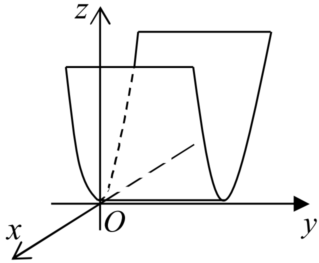
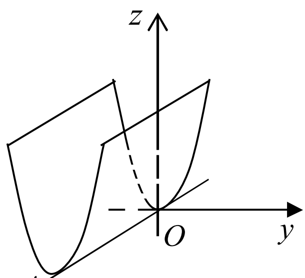
Continued Table B.1

Elliptic cylinder	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	
	$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$	
	$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$	

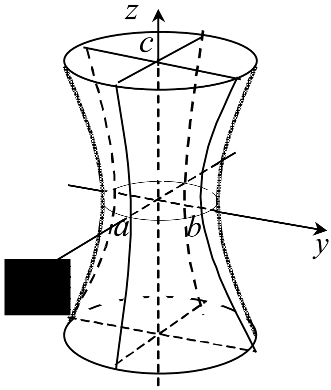
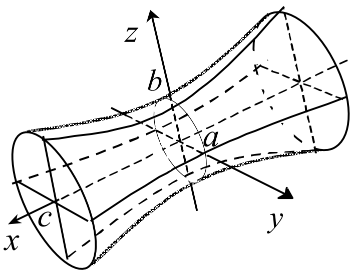
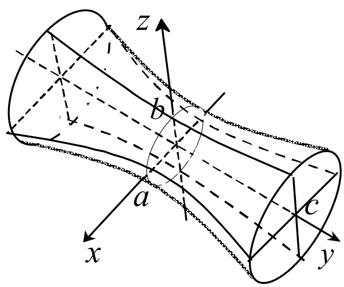
Continued Table B.1

Hyperbolic cylinder	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	
	$\frac{y^2}{a^2} - \frac{z^2}{b^2} = 1$	
	$\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1$	
Parabolic cylinder	$y^2 = 2px$	

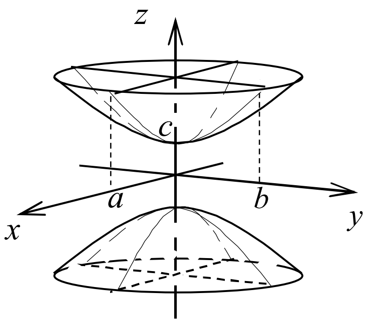
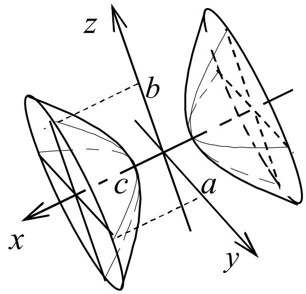
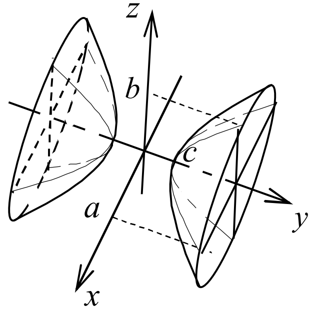
Continued Table B.1

Parabolic cylinder	$x^2 = 2py$	
	$z^2 = 2px$	
	$x^2 = 2pz$	
	$y^2 = 2pz$	

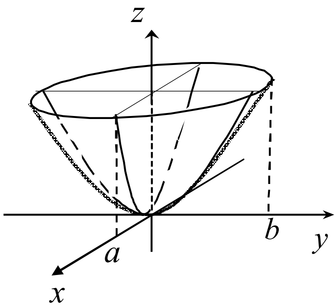
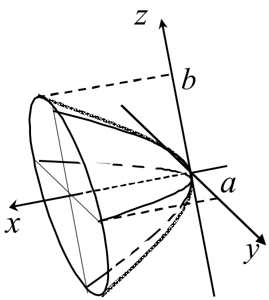
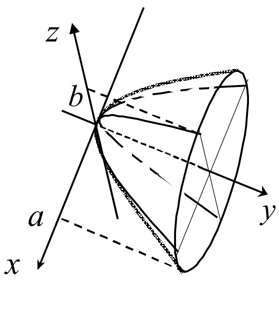
Continued Table B.1

Single-cavity hyperboloid	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$	
	$$\frac{y^2}{a^2}+\frac{z^2}{b^2}-\frac{x^2}{c^2}=1$$	
	$$\frac{x^2}{a^2}+\frac{z^2}{b^2}-\frac{y^2}{c^2}=1$$	

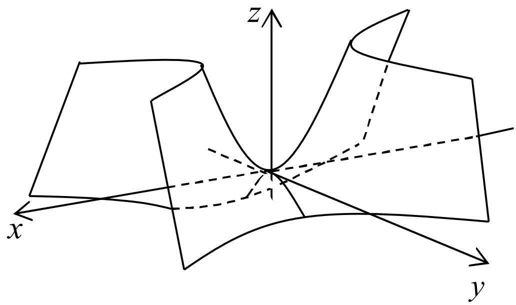
Continued Table B.1

Double-cavity hyperboloid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$	
	$\frac{y^2}{a^2}+\frac{z^2}{b^2}-\frac{x^2}{c^2}=-1$	
	$\frac{x^2}{a^2}+\frac{z^2}{b^2}-\frac{y^2}{c^2}=-1$	

Continued Table B.1

Elliptical paraboloid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$	
	$\frac{y^2}{a^2}+\frac{z^2}{b^2}=x$	
	$\frac{x^2}{a^2}+\frac{z^2}{b^2}=y$	

Ending Table B.1

Hyperbolic paraboloid	$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$$	

APPENDIX C

Frequently used trigonometric formulas

$$\sin^2\alpha + \cos^2\alpha = 1,$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \quad \cot\alpha = \frac{\cos\alpha}{\sin\alpha},$$

$$1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}, \quad 1 + \cot^2\alpha = \frac{1}{\sin^2\alpha},$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha,$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}, \quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2},$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2},$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

APPENDIX D

Table D.1 – The values of trigonometric functions

Value of angle α		Functions			
degrees	radians	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
0°	0	0	1	0	не існує
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	does not exist $(+\infty)$	0
180°	π	0	-1	0	does not exist $(+\infty)$
270°	$\frac{3\pi}{2}$	-1	0	does not exist $(-\infty)$	0
360°	2π	0	1	0	does not exist $(-\infty)$

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